Modular Form

Goals: ① Basic notions (informal)

② There are many basic open questions

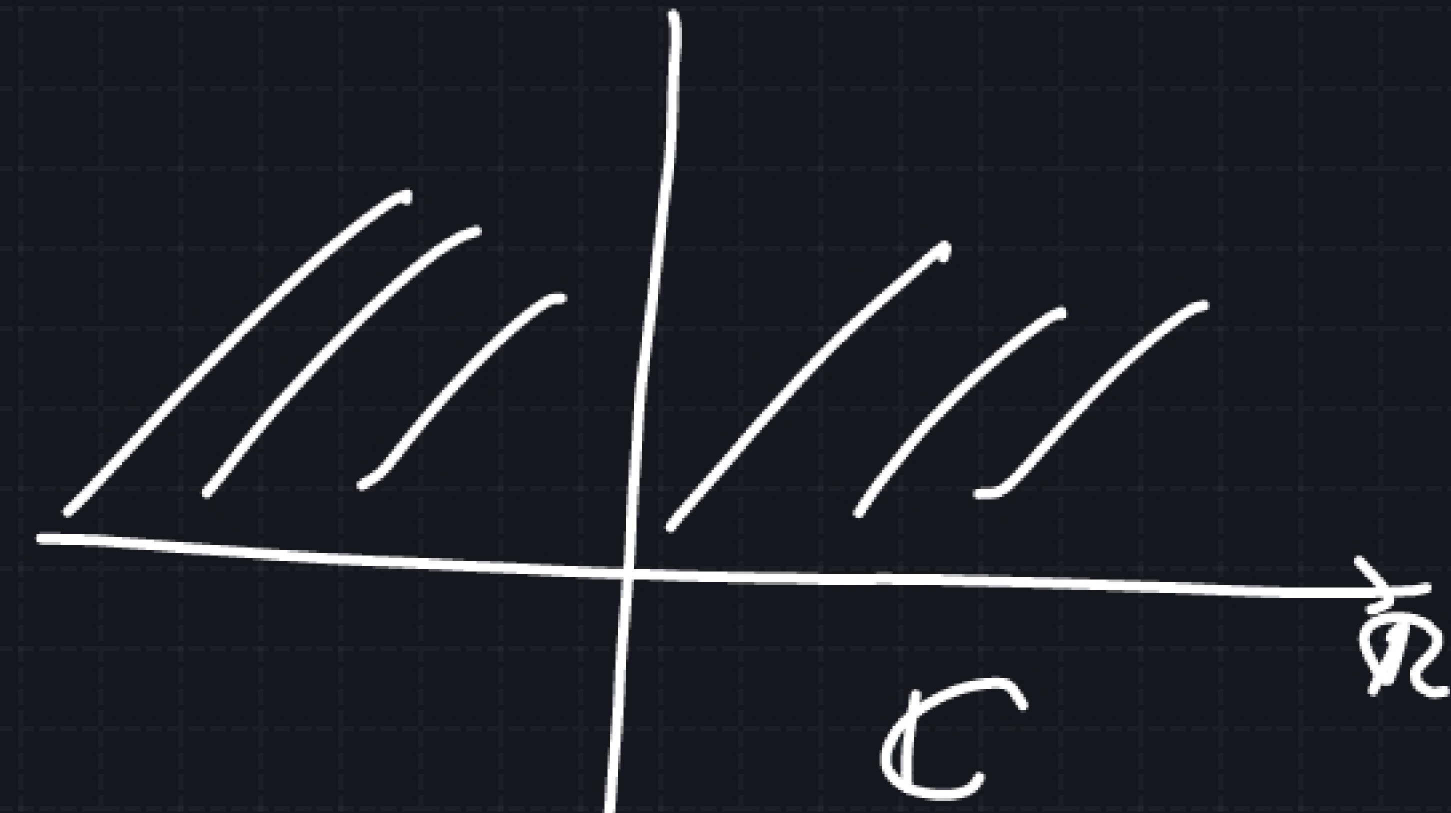
③ Experiments with MF (with computations)

Python (Sage)

12 GB

Cloud Computing

$\mathfrak{h} = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$



$\mathbb{H}$

$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{array} \right\}$

acts on $\mathfrak{h}$:

$$\underset{\underset{SL(2,\mathbb{R})}{\cap}}{(A}, \underset{\underset{\mathfrak{h}}{\in}}{z}) = Az = \frac{az+b}{cz+d} \underset{\underset{\text{exercise}}{\uparrow}}{\in} \mathfrak{h}$$

fractional (or Möbius) transf.

(Lemma: $\mathrm{Im}(Az) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$)

Action: $SL(2,\mathbb{R}) \times \mathfrak{h} \to \mathfrak{h}$

$(A, z) \mapsto Az$

+ (i) $\underset{\underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\uparrow}}{\mathbb{1}} z = z$ (ii) $A(Bz) = (AB)z$ check it

$\mathfrak{h}$ provides a model for hyperbolic geometry:

Point : element of $\mathfrak{h}$

Line : a half circle perp. to $\mathbb{R}$

These two notions satisfy all the axioms of Euclidean geometry except for one: the parallel axiom.

Transfor gp.		discrete subgps	Artists
$O(2) \ltimes T$	Eucl. Geom.	17 (5)	Muslim inner arch
$SL(2,\mathbb{R})$	hyp. Geom.	infinitly many	Escher

discrete subgps of $SL(2,\mathbb{R})$: ~~(A)~~

quaternion gps: Q quaternion algebra $/\mathbb{Q}$, tr, $n()$,
trace norm

is indefinite

$\mathcal{O}$ order in Q

$G := \{ a \in \mathcal{O} \mid n(a) = 1 \} \xhookrightarrow{\text{subgp.}} SL(2,\mathbb{R})$

$\rightsquigarrow$ max = disc. subgp.

e.g. $Q = \mathbb{Q}^{2\times 2}$, $n() = \det()$

maximal $\mathcal{O} = \mathbb{Z}^{2\times 2}$ $\rightsquigarrow G = \boxed{SL(2,\mathbb{Z})}$

$SL(2,\mathbb{Z})$ max. discrete subgp of $SL(2,\mathbb{R})$

subgps of $SL(2,\mathbb{Z})$ of finite index

$$\Gamma(N) = \{A \in SL(2,\mathbb{Z}) : A \equiv 1 (N)\} \qquad N \in \mathbb{Z}_{>0}$$

<u>Ex</u>: $1 \longrightarrow \Gamma(N) \longrightarrow SL(2,\mathbb{Z}) \xrightarrow[\rho_N]{\text{mod } N} SL(2, \mathbb{Z}/N\mathbb{Z}) \longrightarrow 1$

is an exact sequ.

$$\Rightarrow \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\} \supseteq \Gamma(N)$$

$$= \rho^{-1}\left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right) \supseteq \Gamma(N)$$

congruence subgp of $SL(2,\mathbb{Z})$ is a subgp of the form:

$$\rho_N^{-1}(G) \qquad \underset{\text{subgp.}}{G \subseteq} SL(2, \mathbb{Z}/N\mathbb{Z}) \quad \text{some } N$$

$$\left(= \left\{ A \in SL(2,\mathbb{Z}) \;\middle|\; \exists B \in G : A \equiv B \ (N) \right\} \right)$$

the remaining gps are "non-congruence subgps" they exist.
(first one has index 7)

$$\frac{\#\left\{ \begin{array}{c}\text{congruence subgp}\\ \text{of index } n\end{array} \right\}}{\#\left\{ \begin{array}{c}\text{all subgp. of } SL(2,\mathbb{Z})\\ \text{of index } n\end{array} \right\}} \xrightarrow[n \to \infty]{?} 0$$

(1980's)

$SL(2,\mathcal{O})$ K number field, totally real $\supseteq \mathcal{O}$ ring of integers in K

$SL(2,\mathbb{Z})$ $\mathbb{Q}$ $\supseteq \mathbb{Z}$

Hilbert modular gps

<u>Fact</u> if $K \neq \mathbb{Q}$, then $SL(2,\mathcal{O})$ contains only congr. subgps.

(Serre 1970-ε)

$\cdot\,\infty$

$SL(2,\mathbb{Z}) \circlearrowright \mathfrak{h}$

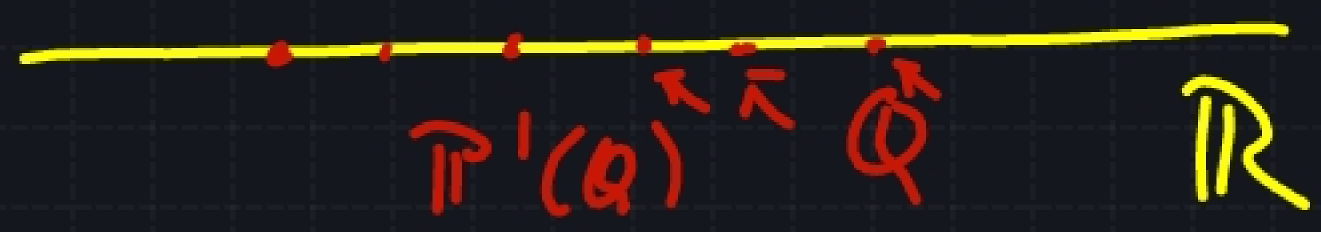
Γ

$SL(2,\mathbb{Z}) \circlearrowright \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty$

$A\tau = \frac{a\tau+b}{c\tau+d}$ formally $\mathbb{Q}^2 \setminus \{0\} / \mathbb{Q}^*$

fand $\overset{\cap}{\mathbb{P}^1(\mathbb{Q})}$

$[a:b] = (a,b)\cdot \mathbb{Q}^*$



$\tau = [a:b] \quad A\tau = [A(a,b)^t]$

$\mathbb{Q} \ni a \leftrightarrow [a:1]$

$\infty := [1:0] = (1,0)\,\mathbb{Q}^*$

Exercise: $SL(2,\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$

$\left(\forall \tau \in \mathbb{P}^1(\mathbb{Q})\ \exists A \in SL(2,\mathbb{Z}) : \tau = A\infty\right)$

Cor: $\#\, {}_{\Gamma}\backslash \mathbb{P}^1(\mathbb{Q}) < \infty$

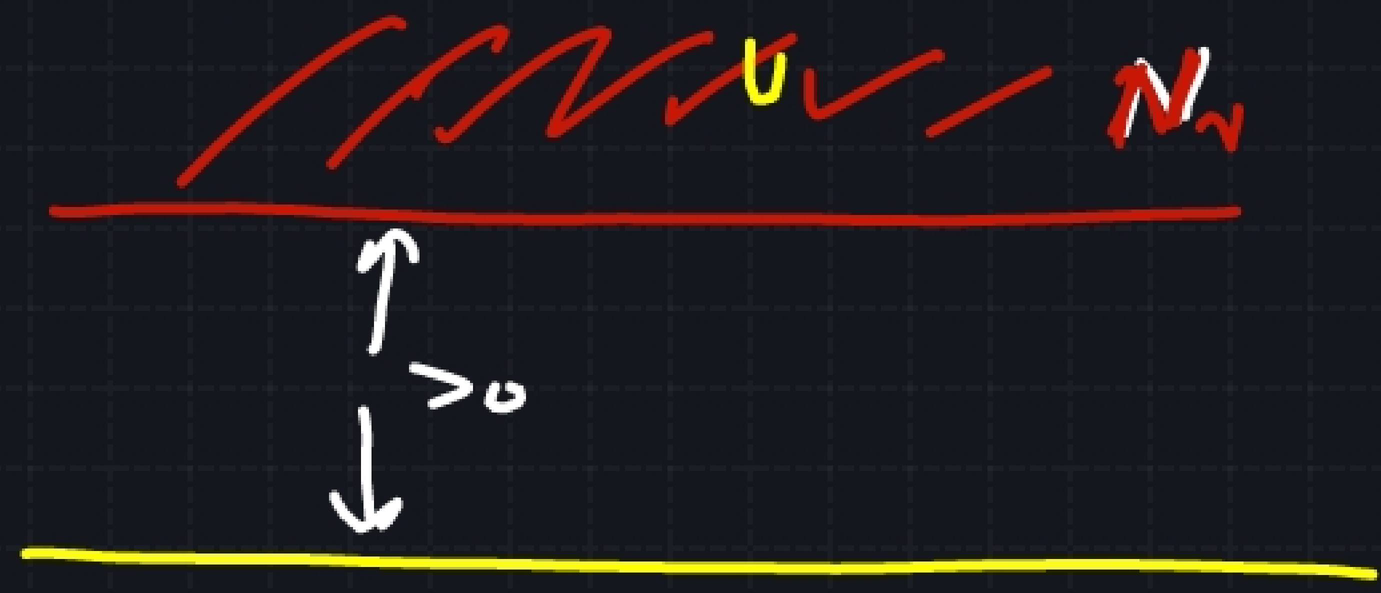
$\Gamma \underset{\text{f. index}}{\subseteq} SL(2,\mathbb{Z})$

cusps

$\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ of Γ

We extend the usual top. on $\mathfrak{h}$ to $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$

neighborhoods of ∞: $N_r := \{ z \in \mathfrak{h} \mid \mathrm{Im}(z) > r > 0 \} \cup \{\infty\}$ for some r



neighborhoods of $p = A\infty = A N_r$

$p = \frac{a}{c}$, $A \in SL(2,\mathbb{Z})$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$

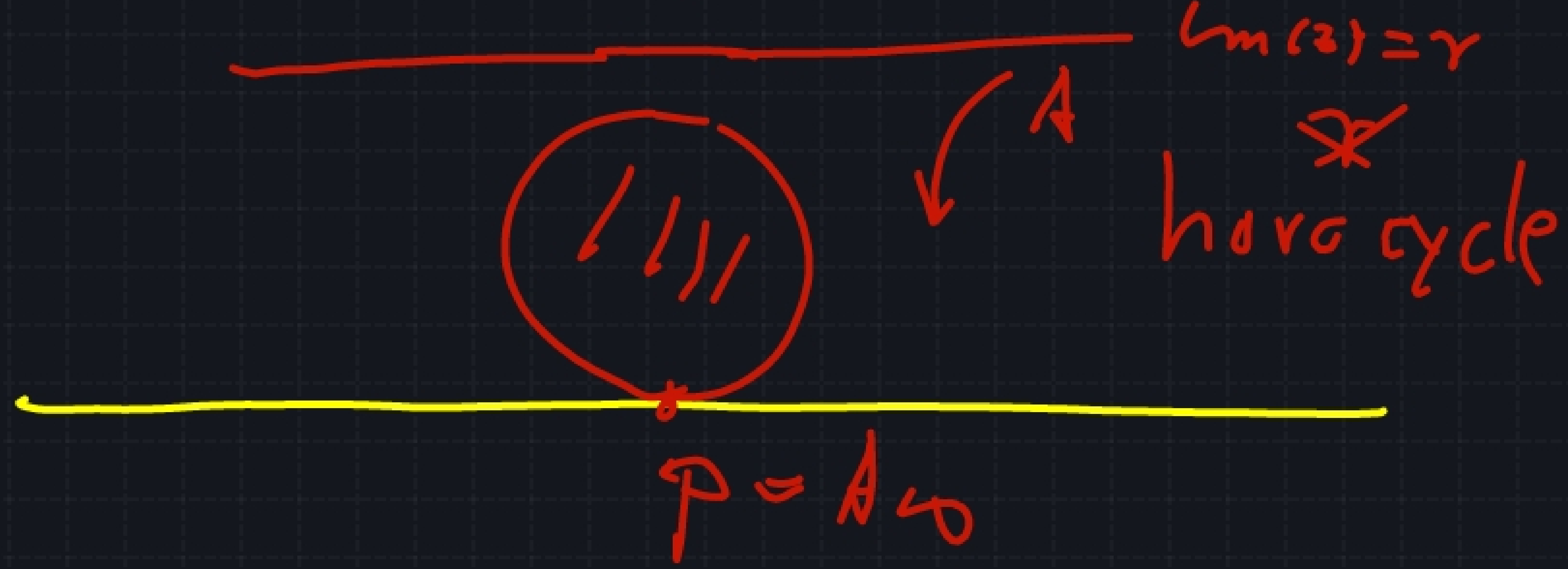
$A N_r = \{ w = Az \mid \mathrm{Im}\, z > r > 0 \} \cup \{p\}$ $\mathbb{R}$

$= \{ z \mid \mathrm{Im}(A^{-1} z) > r > 0 \} \cup \{p\}$

$= \{ z \in \mathfrak{h} : \frac{\mathrm{Im}(z)}{|cz - a|^2} > r > 0 \}$

= interiors of horocycles at p

$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$



$X_\Gamma := \Gamma \backslash \mathfrak{H}^* \quad \left(= \Gamma\backslash \mathfrak{H} \cup \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \right) \quad \Gamma \leq SL(2,\mathbb{Z})$ f. ind.

quotient topology / compact

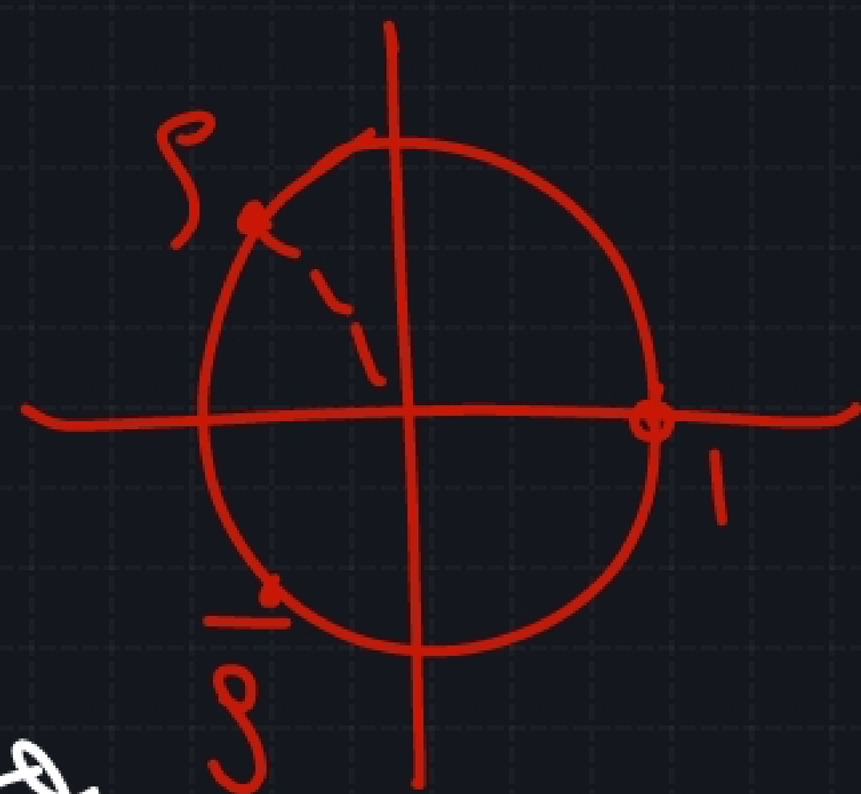
finitely many cusps of Γ

g_Γ even a Riemann surface

$g(\Gamma) :=$ genus of $X_\Gamma = \Gamma\backslash\mathfrak{H}^*$

z, w Γ-equiv. iff $\exists A \in \Gamma : Az = w$

$$g(SL(2,\mathbb{Z})) = 1 + \frac{1}{12} - \frac{1}{2} - \frac{1}{4} - \frac{1}{3} = 0 \checkmark$$



Thm. Assume $-1 \in \Gamma$

$$g(\Gamma) = 1 + \frac{[SL(2,\mathbb{Z}) : \Gamma]}{12} - \frac{\nu_\infty}{2} - \frac{\nu_i}{4} - \frac{\nu_\rho}{3}$$

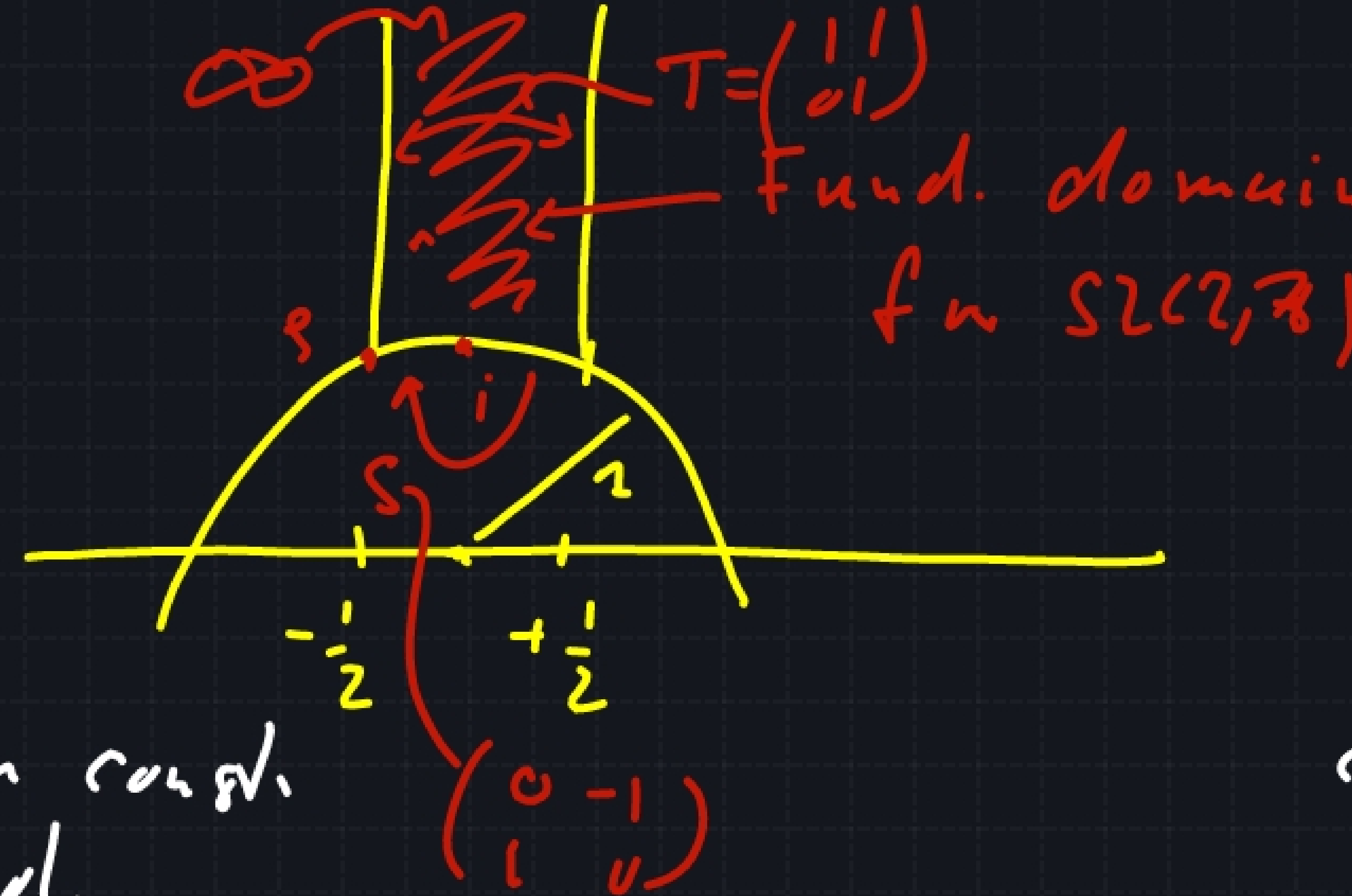
$\nu_\infty = \#$ cusps of $\Gamma = \# \, \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$

$\nu_i =$ Γ-inequivalent fixed pts in $\mathfrak{H}$ that are $SL(2,\mathbb{Z})$-equiv. to i

$= \# \, \Gamma \backslash SL(2,\mathbb{Z}) i$

$\nu_\rho = \ldots \quad \rho \leftrightarrow i \qquad \rho = e^{2\pi i/3}$

$$X_{SL(2,\mathbb{Z})} \simeq \mathbb{P}^1(\mathbb{C})$$



<u>pf</u> of the genus formula

Consider $f : X \xrightarrow{n:1} Y$ non const. hol.

$$\deg \operatorname{div}(\underbrace{f^*\omega}_{\omega\circ f}) = 2g_X - 2$$

$\deg f = n$

ω differential form (hol.)

$$\deg \operatorname{div}(\omega) = 2 - 2g_Y$$

hence

$$2g_X - 2 \overset{!}{=} n \cdot (2g_Y - 2) + \sum_{P \in Y} (e_P - 1)$$

ramification index of f at P

Riemann-Hurwitz formula

To get our genus formula we apply R-H-f. to

$$X_\Gamma \xrightarrow[\text{proj.}]{\pi \text{ canonical}} X_{SL(2,\mathbb{Z})}$$

$$\Gamma \cdot z \longmapsto SL(2,\mathbb{Z})\cdot z \qquad \cong \; \mathbb{P}^1(\mathbb{C}), \text{ genus } 0$$

is ramified only over

$SL(2,\mathbb{Z})\cdot i$

$SL(2,\mathbb{Z})\cdot \rho$

$SL(2,\mathbb{Z})\cdot \infty$

$\deg \pi = [SL(2,\mathbb{Z}) : \Gamma]$, rami

$$2g_{X_\Gamma} - 2 = [SL(2,\mathbb{Z}):\Gamma]\cdot \underbrace{(-2)}_{(2g_{SL(2,\mathbb{Z})}-2)} + \text{correction term} \quad \text{---} \quad \square$$

Def. A modular form of (integer) weight k on Γ ($\underset{\text{f. index}}{\subseteq} SL(2,\mathbb{Z})$) is a hol. map: $f : \mathfrak{H} \longrightarrow \mathbb{C}$

such that:

① $f(Az)(cz+d)^{-k} = f(z)$ for $A \in \Gamma$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

② For all $B \in SL(2,\mathbb{Z})$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and all $r > 0$:

$$\left| f(Bz)(cz+d)^{-k} \right| \text{ bounded for } \operatorname{Im}(z) \geq r.$$