Thm: $M_*(\Gamma)$ is a free M_*-module

$$M_*(\Gamma) = \bigoplus_k M_k(\Gamma) \qquad M_* = \bigoplus_k M_k(SL(2,\mathbb{Z}))$$

$\boxed{\text{Eichler–Zagier: Jacobi forms}}$

Thm Let $A = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_k$ be a M_*-module. Assume

(1) $\dim A_k = O(k) \quad (k \to \infty)$

(2) For any $a, b \in A$ s.t. $E_4 \cdot a = E_6 \cdot b$ there exists $c \in A$ s.t. $a = E_6 c$, $b = E_4 c$.

Then A is free of finite rank over M_*.

$\left(\dim M_k(\Gamma) \sim \frac{k}{12} [SL(2,\mathbb{Z}) : \Gamma \cdot \langle \pm 1 \rangle] \right)$

$\left(\text{For } M_k(\Gamma) \text{ is O.k. since zeroes of } E_4 : SL(2,\mathbb{Z}) \cdot \rho,\ E_6 : SL(2,\mathbb{Z}) \cdot i \right)$

$$\sum_{k \in \mathbb{Z}} M_k(\Gamma) \underset{\text{subspace}}{\overset{\infty\text{-dim.}}{\subseteq}} \mathrm{Hol}(\mathfrak{H})$$

Lemma A $\quad \overset{\uparrow \text{Lemma}}{=} \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma)$

$$M_* := \bigoplus M_k(SL(2,\mathbb{Z})) \underset{\mathbb{C}\text{-alg.}}{\cong} \mathbb{C}[X,Y] \qquad \begin{matrix} E_4, E_6 \\ \uparrow \\ X, Y \end{matrix}$$

Thm. The $\mathbb{C}$-alg. $M_*(\Gamma)$ is finitely generated.

(as generators of this alg. you can take the gen. of $M_*(\Gamma)$ as M_*-module)

Thus: there exist $f_1, \dots, f_d \in M_*(\Gamma)$ (w.l.o.g. modular fcns) such that

$$\begin{aligned} \mathbb{C}[X_1, \dots, X_d] &\longrightarrow M_*(\Gamma) \quad \text{is surjective.} \\ P &\longmapsto P(f_1, \dots, f_d) \end{aligned}$$

Kernel is a ("homogeneous") ideal in $\mathbb{C}[X_1, \dots, X_d]$

and $M_*(\Gamma) = \mathbb{C}[X_1, \dots, X_d]/I$

$\Gamma_0(N)$ $1 \leq N \leq 100$ Find gen. for $M_*(\Gamma_0(N))$

✓

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say $-1 \in \Gamma$.

How would you find generators:

$\mathcal{G}_2$ = basis of $M_2(\Gamma)$, $M_4(\Gamma)' :=$ space spanned by all $g_i \cdot g_j$ ($g_i \in \mathcal{G}_2$)

$\mathcal{G}_4 = \mathcal{G}_2 \cup$ a basis for a complement of $M_4(\Gamma)'$ in $M_4(\Gamma)$

$\mathcal{G}_2 \subset \mathcal{G}_4 \subseteq \mathcal{G}_6 \subseteq \mathcal{G}_{10}$

$\mathcal{G}_6 = \mathcal{G}_4 \cup$ a basis for a complement of $M_6(\Gamma)'$ (span of product from "before") in $M_6(\Gamma)$ ...

stop

$(-1 \in \Gamma)$

Generators of $M_*(\Gamma)$ have wts. ≤ 10

not at all unique

however: $\nu_d :=$ # generators of weight d $(d = 2, 4, 6, 8, 10)$

is indep. of the choice of generators.

<u>Problem.</u> Given Γ $(\ni -1)$. What is $\nu_2, \nu_4, \nu_6, \nu_8, \nu_{10}$?

What is known (work of sev. people $\approx$ 2017)

! wts of generators is bounded by 6. !

$\nu_2 \quad \nu_4 \quad \nu_6$???

$M_*(\Gamma)$ a module over M_*.

We know: $M_*(\Gamma) = M_{k-w_1} f_1 \oplus M_{k-w_2} f_2 \oplus \cdots \oplus M_{k-w_d} \cdot f_d$

$w_1 = wt(f_1)$, $w_2 = wt(f_2)$

$$H_\Gamma = \sum_{k \geq 0} \dim M_k(\Gamma) X^k = \sum_{j=1}^{d} \sum_{k} (\dim M_{k-w_j}) X^k$$

$$= \sum_{j=1}^{d} X^{w_j} \sum_{\ell} \dim M_\ell X^\ell = \frac{\sum_j X^{w_j}}{(1-X^4)(1-X^6)}$$

$$\| \qquad M_* = \mathbb{C}[E_4, E_6] \Rightarrow$$

$$\frac{P_\Gamma(X)}{(1-X^4)(1-X^6)}$$

$$P_\Gamma(X) = \sum_j X^{w_j}$$

$$P_\Gamma = 1 + a_1 X + a_2 X^2 + \cdots + a_{11} X^{11}$$

a_j = # of generators among $f_1, \ldots, f_d$ whose wt. is j

Ex $P_{\Gamma_0(2)} = 1 + X^2 + X^4$

Thus 1 gen. has wt 2, the other one wt 4 (and the const. f. 1)

$M_2(\Gamma) = \mathbb{C}\left(\underbrace{E_2(z) - 2E_2(2z)}_{A}\right)$ $\quad M_*(\Gamma_0(2)) = M_* A + M_* E_4(2z)$

1-dim.

$E_4(z)$

Ex $P_{\Gamma_0(3)}$

$M_4(\Gamma) = \mathbb{C}\cdot A^2$

2-dim.

$E_4(2z) = 1 + 240 q^2 + \dots$

$= 1 + X^2 + X^4 + X^6$

$\delta_2 A$ $\qquad A^2, E_4(2z)$ l. ind.

Find generators

$1 + \underset{0}{\#} q + \dots$

? how to describe f.i. subgroups of $SL(2,\mathbb{Z})$

Let G be a gp., subgp. of finite index n

trans. G-sets of order n with. base pt. $\longrightarrow$ $\left\{ \begin{array}{c} \text{subgp. of index } n \\ \text{in } SL(2,\mathbb{Z}) \end{array} \right\}$

$(X, x_0) \longmapsto \Gamma := \mathrm{Stab}_G(x_0) = \{ g \in G : x_0 g = x_0 \}$

$X \circlearrowleft G \quad \# X = n \quad \leftarrow G\text{-set}$

action

x_0 a fixed el. in X

(orbit-stab.: $\# X = \# \Gamma \backslash G$)

this map is surjective

Indeed: if $\Gamma \underset{n}{\subseteq} G$, then $X = \Gamma \backslash G \circlearrowleft G$

$x_0 = \Gamma$

<u>hence</u>

$$\left\{ \begin{matrix} \text{trans. } G\text{-sets} \\ \text{with base pt.} \\ \text{order } n \end{matrix} \right\} \Big/ \sim \quad \xrightarrow{\text{surj.}} \quad \left\{ \Gamma \underset{n}{\subseteq} G \right\}$$

up to "$\sim$"

$X, x_0 \quad Y, y_0 \quad \mapsto$ same Γ

iff $\exists$ G-equiv. map $\varphi : X \to Y$

$x_0 \mapsto y_0$

$X = \{1, \ldots, n\}$

$x_0 = 1$ / but then a G-action on X is nothing else

$\rho : G \xrightarrow{\text{hom}} S_n$ symm. gp.

s.t. $\rho(G)$ is transitive subgp of S_n

$$\left\{ \begin{array}{c} \rho : G \xrightarrow{\text{hom.}} S_n \\ \rho(G) \text{ trans.} \end{array} \right\} \Big/ \sim \quad \xrightarrow[\text{bij.}]{\sim} \quad \{ \Gamma \leq_n G \}$$

$\underline{\rho}$ $\sim \rho'$ iff $\exists\, \pi \in S_n$: $\pi \circ \rho \circ \pi^{-1} = \rho'$, $1^{\pi} = 1$.

<u>if G is finitely presented</u> this gives us an explicit desc. of all $\Gamma \leq_n G$

e.g. $G = PSL(2, \mathbb{Z}) = SL(2,\mathbb{Z}) / \langle \pm 1 \rangle$

$$G = [\underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\underset{\shortparallel}{S}}, \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}{\underset{\shortparallel}{R}} ; S^2 = R^3 = 1] \left(= \left(\mathbb{Z}/2\mathbb{Z} \right) \underset{\text{free amalgam}}{*} \left(\mathbb{Z}/3\mathbb{Z} \right) \right)$$

$$\left\{ \begin{array}{c} \rho : PSL(2,\mathbb{Z}) \longrightarrow S_n \\ \text{with tran.} \\ \text{image} \end{array} \right\} \Big/ \sim$$

$$\text{bij.} \updownarrow \approx$$

$$\left\{ (s,r) \;\middle|\; \begin{array}{l} s, r \in S_n \\ s^2 = 1,\ r^3 = 1 \end{array} \right\} \Big/ \begin{array}{l} S_n\text{-conjug.} \\ \text{leaving } 1 \text{ fix} \end{array} \longleftrightarrow \left\{ \Gamma \underset{n}{\leq} PSL(2,\mathbb{Z}) \right\}$$

$\underbrace{}_{\langle s, r\rangle \text{ trans.}}$

s = product of transp.
r = ___ of cycles of length 3

(e.g. $s = (23)(45) \in S_7$, $r = (145)(672)$)

$$(s, r) \longmapsto \Gamma = \left\{ S^{n_0} R^{m_0} S^{n_1} R^{m_1} \cdots \;\middle|\; 1^{s^{n_0} r^{m_0} \cdots} = 1 \right\}$$

Schreier-coset graphs to $\Gamma \leq_n G$:

pts. $\Gamma \backslash PSL(2,\mathbb{Z})$ n ptr.

edges: $x \to y$ if $y = xS$

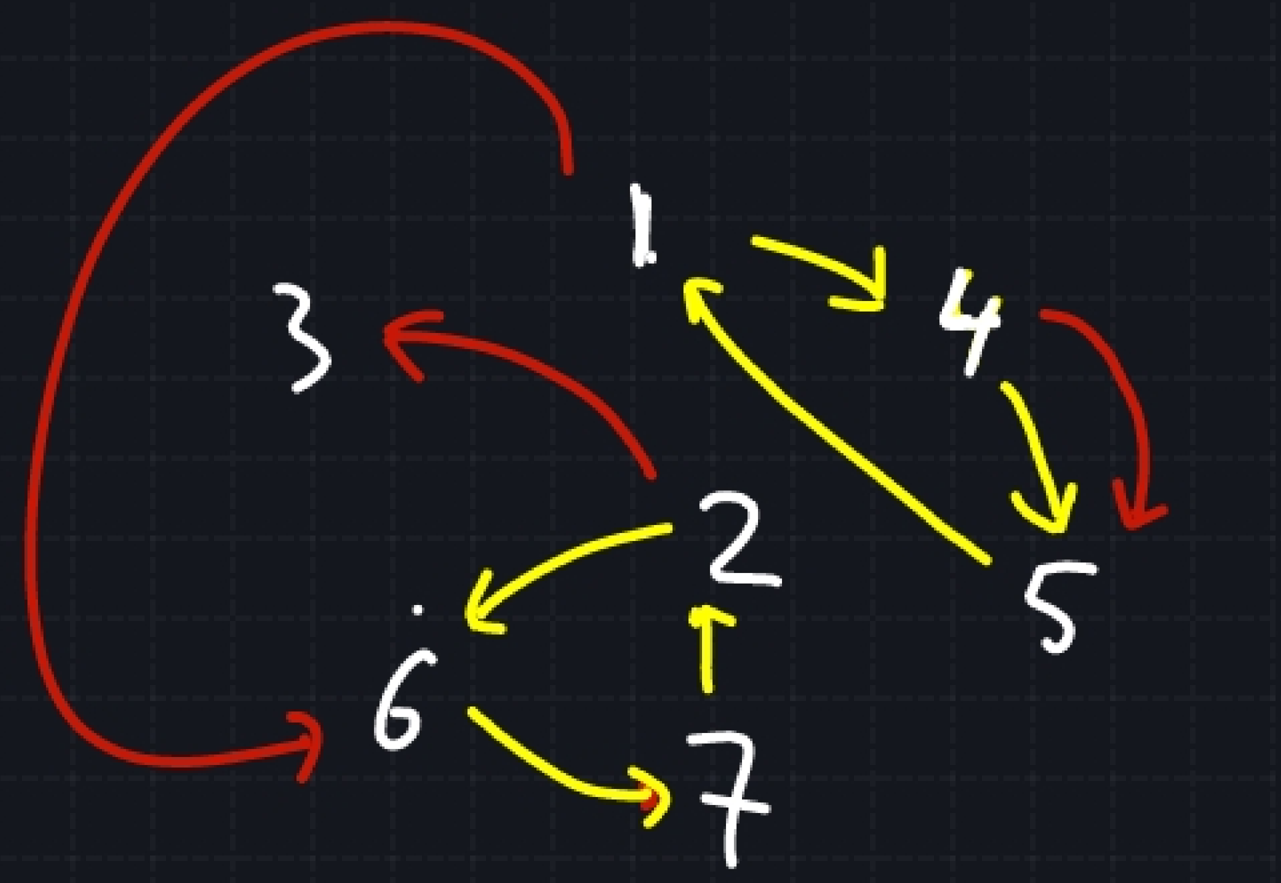
$x \to y$ if $y = xR$

$SL(2,\mathbb{Z})$

$= \left[S, R \;\middle|\; \begin{array}{l} S^4 = 1 \\ S^2 = R^3 \end{array} \right]$

$s = (23)(45)(16) \in S_7$

$r = (145)(672)$



$$\Gamma = \left\{ s^{n_0} r^{m_0} \cdots \;\middle|\; s^{n_0} r^{m_0} s^{n_1} r^{n_1} = 1 \right\}$$

$\xleftrightarrow{1-1}$ cycles on the graph starting + ending in 1