(1) $f(Az)(cz+d)^{-k} =: (f|_k A)(z)$ slash-operator

$\forall A \in \Gamma : \quad f|_k A = f \quad$ (prop. (1))

$\Gamma \subseteq SL(2,\mathbb{Z})$ finite index. $\Longrightarrow \exists n \in \mathbb{Z}_{>0} : \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \in \Gamma$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z = z+n$

hence with such n: $f \Big| \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = f$, i.e. $f(z+n) = f(z)$

hence $f = \sum_{\ell \in \mathbb{Z}} a(\ell)\, q^{\frac{\ell}{n}}$ with suitable $a(\ell) \in \mathbb{C}$

$q(z) = e^{2\pi i z}$

(2) $\Longrightarrow a(\ell) = 0$ unless $\ell \geq 0$

Similarily, for any $B \in SL(2,\mathbb{Z})$

①, ② (+ "finite index") imply that

$$f|_k B = \sum_{\ell \geq 0} a_B(\ell)\, q^{\ell/n_B} \qquad \text{expansion at } B\infty$$

Space of Modular form wt k, on Γ: $\quad M_k(\Gamma)$

Def. $f \in M_k(\Gamma)$ is called cusp form, if

$\forall B \in SL(2,\mathbb{Z})$: $a_B(0) = 0$.

$f \in M_k(\Gamma)$: $f = \sum_{\ell \geq v} a(\ell)\, q^{\ell/n}$ (a certain n; if $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then we can choose $n = 1$)

$G \curvearrowright V$, $V^G = \{ v \in V \mid g \cdot v = v \;\forall g \in G \}$

if G is finite: Choose $v_0 \in V$: $\sum_{g \in G} g \cdot v_0 \in V^G$ ("tracing")

Choose any hol. $F : \mathfrak{H} \to \mathbb{C}$, e.g. $F \equiv 1$

$$\sum_{A \in \Gamma} F|_k A \; ; \; f(z) := \sum_{A \in \Gamma} (1|_k A)(z) = \sum_{\substack{A \in \mathrm{Stab}_\Gamma(1) \backslash \Gamma \\ \begin{pmatrix} * & * \\ c & d \end{pmatrix}}} \frac{1}{(cz+d)^k}$$

$\mathrm{Stab}_\Gamma(1) = \langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rangle$

converges indeed against a MF if $k \gg 0$ (more prec. if $k \geq 3$)

"Eisenstein series"

For $SL(2,\mathbb{Z})$: $E_4(z) := * \sum_{\substack{c,d \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^{4\,6}} \in M_4(SL(2,\mathbb{Z}))$

E_6

One obtains:

$$E_4 = 1 + 240 \sum_{\ell \geq 1} \sigma_3(\ell)\, q^\ell$$

$$\sum_{d \mid \ell} d^3$$

$$E_6 = 1 - 504 \sum_{\ell \geq 1} \sigma_5(\ell) q^\ell \in M_6(SL(2,\mathbb{Z}))$$

(Serre: A course in arithm.)

Miyake: MF

Lang: MF

Shimura: MF

For $k=2$ (with Hecke trick)

$$E_2^* := 1 - 24 \cdot \sum_{\ell \geq 1} \sigma_1(\ell) q^\ell - \frac{3}{\pi \operatorname{Im}(z)} : \quad E_2^* \big|_2 A = E_2^* \quad \forall A \in SL(2,\mathbb{Z})$$

nonhol.

$\in M_2(SL(2,\mathbb{Z}))$

Exercise: (i), (ii) SeA $\quad \eta(z) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n).$

Then one has:

$$\frac{\eta'}{\eta} \overset{(i)}{=} \frac{\pi i}{12} E_2 \qquad \left(\frac{\eta'}{\eta} = \frac{d}{dz} \log \eta = 2\pi i \left(\frac{1}{24} - \sum_{n\geq 1} \frac{n q^n}{1-q^n} \right) = \ldots 2\pi i \, E_2 \right)$$

exercise

24-th root of unity

$$\overset{(ii)}{\Rightarrow} \quad \eta(Az) = \zeta_{24} \sqrt{cz+d} \; \eta(z)$$

$\forall \; A \in SL(2,\mathbb{Z})$

transf. for E_2^* $\rightsquigarrow$ transf. for E_2

Hence

$$\Delta := \eta^{24} \in M_{12}^{\text{cusp}}(SL(2,\mathbb{Z}))$$

Thm. Every $f \in M_k$ $(:= M_k(SL(2,\mathbb{Z})))$ is a polynomial in E_4, E_6.

in particular: $M_4 = \mathbb{C}\cdot E_4$, $M_6 = \mathbb{C}\cdot E_6$, $M_8 = \mathbb{C}\cdot E_4^2$, $M_{10} = \mathbb{C}\cdot E_4 E_6$

$$M_{12} = \mathbb{C} E_4^3 + \mathbb{C} E_6^2, \cancel{\Delta} \quad ??? \quad \left(\Delta = \frac{E_4^3 - E_6^2}{\underbrace{1728}_{12^3}} \right)$$

$$= \mathbb{C}\cdot E_4^3 + \mathbb{C}\Delta$$

Pf. $f \in M_k$, $f = a(0) + a(1)q + \dots$, for suit. $a, b \in \mathbb{Z}_{\geq 0}$: $E := E_4^a E_6^b$

then $f - E$ is cusp form $= *q + *q^2 + \dots$. Since Δ has no zeros in $\mathfrak{h}$: $\in M_k$

$\frac{f - E}{\Delta} \in M_{k-12} \rightsquigarrow$ inductively: $f = \mathrm{pol}(E_4, E_6)$. $\square$

! M_k is of finite dimension

Thm. E_4 and E_6 are algebraically independent.

In particular:

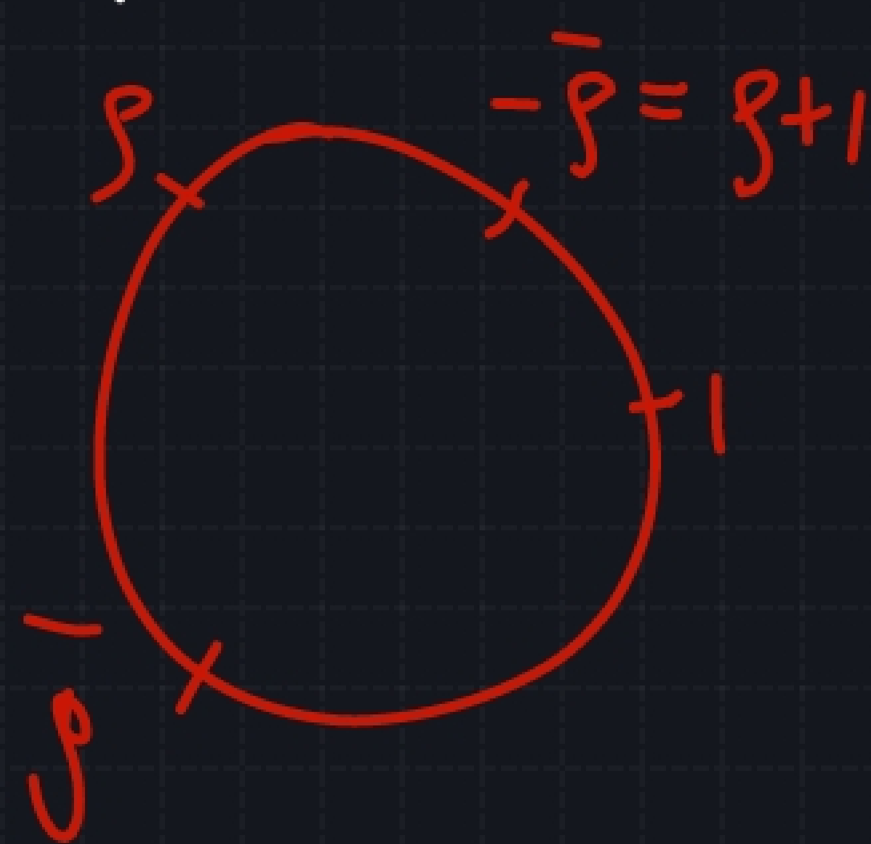
$$\mathbb{C}[X,Y] \longrightarrow M_* := \bigoplus_{k\in\mathbb{Z}} M_k$$

$$P \longmapsto P(E_4,E_6)$$

is an isomorphism of $\mathbb{C}$-algebras.

Serre

Valence formula: counts how many zeros a MF has



Valence formula: E_4 has ρ as only zero (up to $SL(2,\mathbb{Z})$-translates)

transform. for $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$$E_4\left(-\frac{1}{z}\right) z^{-4} = E_4(z)$$

Set $z=\rho := e^{2\pi i/3}$

$$E_4(-\bar\rho)\,\underbrace{\rho^{-4}}_{\neq 1} = E_4(\rho)$$

$-\bar\rho = \rho+1$

hence

$E_4(\rho+1) = E_4(\rho)$ $\Rightarrow$ $E_4(\rho)=0$

Similarily (exercise): $E_6(i) = 0$

and i is in fact the only zero (up to $SL_2(\mathbb{Z})$-translates).

E_4, E_6 alg. indep.

Otherwise there exist $P \neq 0$ s.t. $P(E_4, E_6) = 0$

w.l.o.g. P has minimal degree among all non-zero pols annihilating E_4, E_6. Write: $P = a X^n + \tilde{P} + b Y^m$

$\tilde{P} = XY\hat{P}$ no other terms since P has degree, say k where $\deg(X) = 4$, $\deg(Y) = 6$

$$0 = P(E_4, E_6) = a E_4^n + E_4 E_6 \hat{P}(E_4, E_6) + b E_6^m$$

$z = \rho$: $\Rightarrow b = 0$

$z = i$: $\Rightarrow a = 0$

$\Rightarrow \hat{P}(E_4, E_6) = 0$ ↯ to the min. of $\deg(P)$.

Interesting subgps: $\Gamma_0(N)$ $\left(= \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SL(2,\mathbb{Z})\right)$

Construction methods:

$$E_2^*(z) - N E_2^*(Nz) \in M_2(\Gamma_0(N))$$

$$\left(-\frac{3}{\pi \operatorname{Im}(z)} - N \frac{-3}{\pi \operatorname{Im}(Nz)} \right) = 0$$

$f \in M_k$

$f(Nz) \in M_k(\Gamma_0(N))$

exercise

$$\sum_{t|N} c_t E_2(tz) \in M_2(\Gamma_0(N)) \quad \text{if } \sum_{t|N} c_t \, t = 0$$

$$E_2^*(z) - N E_2^*(Nz) \underset{\text{exercise}}{=} 1 - N - 24 \sum_{\ell \geq 1} q^\ell \left(\sum_{\substack{d|\ell \\ N \nmid d}} d \right)$$

E_2 has a transf. formula (from the one for E_2^{γ}):

$$E_2\Big|_2 A = E + *(cz+d)^{-1} \qquad \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\right)$$

Let $f \in M_k(\Gamma)$:

$$\delta_k f := \frac{1}{2\pi i} f' - \frac{k}{12} E_2 f \in M_{k+2}(\Gamma)$$

modular derivative (Serre derivative)

$f \in M_k(\Gamma) \qquad g \in M_h(\Gamma)$

$$[f,g]_n := \frac{1}{(2\pi i)^n} \sum_{\substack{r,s \\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{h+n-1}{r} f^{(r)} g^{(s)} \in M_{k+h+2n}(\Gamma)$$

Rankin-Cohen bracket

$$M_*(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma)$$

sum is direct (exercise)

graded $\mathbb{C}$-algebra

Remark: interesting structure:

$\mathbb{C}[E_2, E_4, E_6]$ is closed under $\frac{1}{2\pi i}\frac{d}{dz}$

(1-2-3 modular forms Zagier)

In fact: $M_k(\Gamma)$ is finite dim., $M_k(\Gamma)=0$ if $k<0$, $M_0(\Gamma)=\mathbb{C}$.

Thm. For all k, one has

$$\dim M_k(\Gamma) - \dim S_{2-k}(\Gamma) = \frac{k-1}{12}[SL(2,\mathbb{Z}):\bar\Gamma] - \frac{(k-1)_4}{4}\nu_i - \frac{(k-1)_3}{3}\nu_\rho + \frac{1}{2}\nu_\infty - \frac{(k)_2}{2}\nu_{irr.}$$

$(x)_m = y$ s.t. $-\frac{m}{2} < y \leq +\frac{m}{2}$, $y \equiv x \bmod m$

Note: $(k)_2 = \begin{cases} 1 & k \text{ odd} \\ 0 & k \text{ even} \end{cases} =: a(k)$

$\boxed{k=1 \ \dim M_1(\Gamma) - \dim S_1(\Gamma)}$

$\left(\bar\Gamma = \Gamma\langle\pm 1\rangle\right)$, $\nu_\infty = \#\text{cusps} = \#\,\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$,

$\nu_{irr.} = \#$ irregular cusps., $\nu_i = \#$ of fixed point of Γ mod Γ that are in $SL(2,\mathbb{Z})\cdot i$)

$\rho \qquad \rho$

$p \in \mathbb{P}^1(\mathbb{Q})$ is irregular cusp for Γ if

$$\mathrm{Stab}_\Gamma(p) \underset{\substack{SL(2,\mathbb{Z})\\ -\text{conj.}}}{\sim} \left\langle \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \right\rangle \quad \text{for a suitable } b \in \mathbb{Z}.$$

---

pf. (look in Shimura): it is the Riemann-Roch-theorem applied to certain line bundles on X_Γ whose hol. sections correspond to the MFs in $M_k(\Gamma)$. $\square$

??? $\dim M_1(\Gamma)$ (or $S_1(\Gamma)$)

$\dim M_1(\Gamma) = \#$ 2-dimensional representations ρ (mod equiv.) of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho\begin{pmatrix}\text{complex}\\\text{conjugation}\end{pmatrix} = \begin{pmatrix}-1 & \\ & -1\end{pmatrix}$

$\overline{\mathbb{Q}}$: field of alg. numbers $\subseteq \mathbb{C}$

$f = \sum a_f(n) q^n$, f Hecke eigenform norm. form

$\rightsquigarrow L(f,s) := \sum_{n\geq 1} \frac{a_f(n)}{n^s} = \prod_p \frac{1}{1-a_f(p)p^{-s}+p^{k-1-2s}}$

$=$ since f Hecke eigen.

$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, 2-dim. $\xrightarrow{\text{hom}} GL_2(K)$, $K \supseteq$ f.ext. $\mathbb{Q}_\ell$

$L(\rho,s) = \prod_p \frac{1}{\chi_{\rho(F_p)}(p^{-s})}$

F_p: Frob. elem.

Fact. For every Hecke-eigen, new-form f there exists a ρ s.t.

$$L(f,s) = L(\rho,s)$$

(For $k=1$: $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2,\mathbb{C})$)

??? does the reverse also hold true:

? if ρ is a rep., is there an f s.t. $L(f,s) = L(\rho,s)$?

$k=1$ ✓ in general we do not know: part of the Langlands conj.

$E/\mathbb{Q}$ $\quad L(E,s) = L(\rho,s)$, ρ comes from the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Tate module of E:

$$\varprojlim_n E[\ell^n] = T_\ell(E)$$

2-dim on $\mathbb{Z}/\ell^n\mathbb{Z}$, ℓ prime

2-dim on $\mathbb{Q}_\ell$

Note ⊛ $a(k) - a(k-4) - a(k-6) + a(k-10) = 0 \quad \forall k$

hence Hilbert-Poincaré-series of $M_*(\Gamma)$:

$$H_\Gamma := \sum_{k\geq 0} \dim M_k(\Gamma) X^k = \sum_{k\geq 0} a(k) X^k$$

⊛ can be stated as

$$\Big(\sum a(k) X^k\Big)(1-X^4)(1-X^6) = P_\Gamma(X) \text{ polynomial in } X \text{ of degree} \leq 11$$

$$\boxed{(**)\quad H_\Gamma = \sum_{k\geq 0} \dim M_k(\Gamma) X^k = \frac{P_\Gamma(X)}{(1-X^4)(1-X^6)}}$$

✓ $M_*(T)$ is a M_*-module

$(\ast\ast)$ suggests that $M_*(T)$ is **free** over M_*