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Inventiones mathematicae

Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms

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1. Introduction and discussion

Fix a positive integer m and denote by $J^-_{2,m}$ and $J^+_{2,m}$ ($S^-_{2,m}$ and $S^+_{2,m}$) the space of holomorphic and skew-holomorphic Jacobi (cusp) forms of weight 2 and index m respectively. By definition these are the spaces of smooth functions $\phi(\tau, z)$ in two variables τ, $z \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, which are periodic in τ and z respectively with period 1, which satisfy $\phi\left(\frac{-1}{\tau}, \frac{z}{\tau}\right)e^{-2\pi i m\frac{z^2}{\tau}} = \tau^2\phi(\tau, z)$ if ϕ is a holomorphic Jacobi form, and $\phi\left(\frac{-1}{\tau}, \frac{z}{\tau}\right)e^{-2\pi i m\frac{z^2}{\tau}} = \bar{\tau}|\tau|\phi(\tau, z)$ if ϕ is skew-holomorphic, and the Fourier expansions of which have the form

$$\phi(\tau, z) = \sum_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv r^2 \bmod 4m}} C_\phi(\Delta, r) e^{2\pi i\left(\frac{r^2 - \Delta}{4m}u + \frac{r^2 + |\Delta|}{4m}iv + rz\right)} \qquad (\tau = u + iv)$$

where the coefficients $C_\phi(\Delta, r)$ depend on r only modulo $2m$ and vanish for $\Delta > 0$ ($\Delta \geqq 0$) if ϕ is a holomorphic Jacobi form, and for $\Delta < 0$ ($\Delta \leqq 0$) if ϕ is skew-holomorphic.

Furthermore, we consider integral quadratic polynomials $[a, b, c](x) = ax^2 + bx + c$. The group $SL_2(\mathbb{Z})$ acts on these by $[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x) = [a, b, c]\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)(\gamma x + \delta)^2$. For a pair of integers Δ, r with $\Delta \equiv r^2 \bmod 4m$ define

$$\mathcal{Q}(\Delta, r) := \{[ma, b, c] \mid a, b, c \in \mathbb{Z},\ b^2 - 4mac = \Delta,\ b \equiv r \bmod 2m\}\ .$$

This set is invariant under $\Gamma_0(m) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ m\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SL_2(\mathbb{Z})$. For a fundamental discriminant Δ_0 which is a square modulo $4m$ we denote by $\chi_{\Delta_0}: \{[ma, b, c] | a, b, c \in \mathbb{Z}\} \to \{0, \pm 1\}$ the generalized genus character introduced in [G-K-Z], i.e.

$$\chi_{\Delta_0}([ma, b, c]) = \begin{cases} \left(\frac{\Delta_0}{n}\right) & \text{if } \Delta_0 \text{ divides } b^2 - 4mac \text{ such that } (b^2 - 4mac)/\Delta_0 \\ & \quad \text{is a square modulo } 4m \text{ and } \gcd(a, b, c, \Delta_0) = 1 \\ 0 & \text{otherwise .} \end{cases}$$

Here n is any integer relative prime to Δ_0 and represented by one of the quadratic forms $m_1ax^2 + bxy + m_2cy^2$ with $m = m_1m_2$, $m_1, m_2 > 0$. (If $\gcd(a, b, c, \Delta_0) = 1$ such n, m_1, m_2 exist, and if $\frac{(b^2 - 4mac)}{\Delta_0}$ is a square modulo $4m$ the value $\left(\frac{\Delta_0}{n}\right)$ is independent of the special choices of n or m_1, m_2, cf. [G-K-Z; I.2, Proposition 1.] The function χ_{Δ_0} is $\Gamma_0(m)$-invariant. Finally, for any integral $[a, b, c]$ and any integer $N \neq 0$ set

$$\operatorname{sign}([a, b, c]) = \begin{cases} \operatorname{sign}(a) & \text{if } ac < 0 \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$\varepsilon_N([a, b, c]) = \begin{cases} -\left(\frac{a}{N} - \frac{1}{2}\right) & \text{if } c = 0,\ 0 < a < N \\ +\left(\frac{c}{N} - \frac{1}{2}\right) & \text{if } a = 0,\ 0 < c < N \\ 0 & \text{otherwise .} \end{cases}$$

Note that for each discriminant Δ there are only finitely many integral polynomials $[a, b, c]$ with $b^2 - 4ac = \Delta$ such that $\operatorname{sign}([a, b, c])$ or $\varepsilon_N([a, b, c])$ is different from 0. Moreover, $\operatorname{sign}([a, b, c]) \neq 0$ implies $b^2 - 4ac > 0$, and $\varepsilon_N([a, b, c]) \neq 0$ implies that $b^2 - 4ac$ is a perfect square.

The aim of this paper is to state (and to prove) the following theorem.

Theorem. *For $A \in SL_2(\mathbb{Z})$ and integers Δ_0, r_0 with Δ_0 a fundamental discriminant, $\Delta_0 \equiv r_0^2 \bmod 4m$, define*

$$\phi_{A, \Delta_0, r_0}(\tau, z) = \sum_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv r^2 \bmod 4m}} C_{A, \Delta_0, r_0}(\Delta, r) e^{2\pi i\left(\frac{r^2 - \Delta}{4m} u + \frac{r^2 + |\Delta|}{4m} iv + rz\right)}$$

where

$$C_{A, \Delta_0, r_0}(\Delta, r) = \sum_{Q \in \mathcal{Q}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q)[\operatorname{sign}(Q \circ A) + \varepsilon_{m|\Delta_0|}(Q \circ A)] .$$

Then $\phi_{A, \Delta_0, r_0}(\tau, z)$ defines a holomorphic Jacobi form in $J_{2,m}^-$ if $\Delta_0 < 0$, and a skew-holomorphic Jacobi form in $J_{2,m}^+$ if $\Delta_0 > 0$. Moreover, any Jacobi form from $J_{2,m}^-$ or $J_{2,m}^+$ is obtained as a linear combination of the functions $\phi_{A, \Delta_0, r_0}(\tau, z)$ and of Eisenstein series.

Remarks 1. Note that the sum defining $C_{A,\Delta_0,r_0}(\Delta, r)$ is actually finite since the expressions $\operatorname{sign}(Q \circ A)$ and $\varepsilon_{m|\Delta_0|}(Q \circ A)$ vanish for all but finitely many Q with given discriminant $\Delta\Delta_0$. 2. The term $\varepsilon_{m|\Delta_0|}(Q \circ A)$ is non-zero only for the Δ which are square multiples of Δ_0. 3. $\phi_{A,\Delta_0,r_0}(\tau, z)$ for fixed Δ_0, r_0 depends only on the coset of A in $\Gamma_0(m)\backslash SL_2(\mathbb{Z})$, as is easily deduced from the $\Gamma_0(m)$-invariance of χ_{Δ_0} and $\mathcal{Q}(\Delta\Delta_0, rr_0)$. 4. The Fourier coefficients of the Jacobi Eisenstein series can be explicitly calculated (cf. [E-Z], [S]).

There are various intimate connections between Jacobi forms and elliptic modular forms; via these connections, the stated theorem can also be read as a theorem about elliptic modular forms. The deepest of such connections is due to the fact that Jacobi forms represent, in a sense, a law which combines interesting arithmetical data of elliptic modular forms: Hecke eigenvalues and special values of twisted L-series in the critical strip.

We explain the latter statement in more detail for the case of Jacobi forms of weight 2. There is, for each pair of integers Δ_0, r_0 such that $r_0^2 \equiv \Delta_0 \bmod 4m$ and such that Δ_0 is a fundamental discriminant, a lifting map $\mathscr{S}_{\Delta_0,r_0}$ which associates to a Jacobi form ϕ from $J_{2,m}^-$ or $J_{2,m}^+$ with Fourier coefficients $C_\phi(\Delta, r)$ an elliptic modular form with l-th Fourier coefficient

$$\sum_{a|l} \left(\frac{\Delta_0}{a}\right) C_\phi\left(\Delta_0 \frac{l^2}{a^2}, r_0 \frac{l}{a}\right).$$

If ϕ is a cusp form this modular form is always a cusp form apart from the case $\Delta_0 = 1$. In the latter case it equals a cusp form plus a linear combination of functions $E_2^*(d\tau)$, where d runs through the divisors of m and E_2^* denotes the (non-holomorphic) modular form of weight 2 on $SL_2(\mathbb{Z})$ which is given by

$$E_2^*(\tau) = \frac{-1}{24} + \frac{1}{8\pi \operatorname{Im}(\tau)} + \sum_{l \geqq 1} \left(\sum_{d|l} d\right) e^{2\pi i l \tau}.$$

The maps $\mathscr{S}_{\Delta_0,r_0}$ are Hecke equivariant, their images are contained in a certain natural subspace $\mathfrak{M}_2(m)$ of the space of all elliptic modular forms of weight 2 on $\Gamma_0(m)$ which contains all newforms, and there exist linear combinations of them which define isomorphisms of $J_{2,m}^- \oplus J_{2,m}^+$ with $\mathfrak{M}_2(m)$ (cf. [S-Z], [S]). Combining these statements with our theorem, we find that the functions $\mathscr{S}_{\Delta_1,r_1}(\phi_{A,\Delta_0,r_0})$, where the Δ_i ($i = 0, 1$) are fundamental discriminants with $\Delta_i \equiv r_i^2 \bmod 4m$ and A ranges over $\Gamma_0(m)\backslash SL_2(\mathbb{Z})$, span the space $\mathfrak{M}_2(m)$ (up to possible linear combinations of Eisenstein series). In particular, all cusp-newforms of weight 2 on $\Gamma_0(m)$ can be obtained by this explicit construction.

If $f(\tau) = \sum a(l) e^{2\pi i l \tau}$ is a new- cusp- and Hecke eigenform, and ϕ the Jacobi cusp form in $S_{2,m}^-$ or $S_{2,m}^+$ corresponding to it under the lifting maps $\mathscr{S}_{*,*}$, then for any fundamental discriminant Δ which is a square modulo $4m$ and prime to m one has

$$|C_\phi(\Delta, r)|^2 = \frac{\sqrt{|\Delta|}\langle\phi|\phi\rangle}{2\pi\langle f|f\rangle} L(f, \Delta, 1).$$

Here $L(f, \Delta, s) = \sum_{l \geqq 1} \left(\frac{\Delta}{l}\right) \frac{a(l)}{l^s}$ is the twisted L-series of f, for r one can choose any solution of $r^2 \equiv \Delta \bmod 4m$ and '$\langle\,|\,\rangle$' denotes Petersson scalar product (cf. Sect. 2

for the exact definition in the case of Jacobi forms). This is proved in [G-K-Z; Corollary 1 in Sect. II.4] for holomorphic Jacobi forms but is true for skew-holomorphic ones too (one can copy the proof given in [G-K-Z], using the Proposition and its Corollary in Sect. 2 of this paper, to derive the corresponding result for skew-holomorphic forms.) Thus the stated theorem can then be used to produce 'Tunnell-like' theorems in an algorithmic way, i.e. it can be used to describe explicitly a computable function of fundamental discriminants Δ whose square gives $|\Delta|^{1/2} L(f, \Delta, 1)$ up to a constant independent of Δ.

As an illustration of the theorem we consider the simplest non-trivial case, i.e. the case $m = 11$. The space of modular forms of weight 2 on $\Gamma_0(11)$ is spanned by the Eisenstein series

$$E(\tau) = E_2^*(\tau) - 11E_2^*(11\tau) = \frac{5}{12} + \sum_{l=1}^{\infty} \left(\sum_{11 \nmid d | l} d \right) q^l$$

and the cusp form

$$S(\tau) = \eta(\tau)^2 \eta(11\tau)^2 = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 ,$$

where $q = e^{2\pi i \tau}$. The space $S_{2,11}^+$ contains the 'trivial' cusp form

$$T(\tau, z) := \sum_{r, s \in \mathbb{Z}} (r + 22s) e^{2\pi i \left(\frac{r^2 - (r+22s)^2}{44} u + \frac{r^2 + (r+22s)^2}{44} iv + rz \right)}$$

which satisfies $\mathscr{S}_{1,1}(T) = E$, as can easily be checked. The set $\Gamma_0(11) \backslash SL_2(\mathbb{Z})$ is indexed by elements of $\mathbb{P}^1(\mathbb{Z}/11\mathbb{Z})$ via $d \leftrightarrow \Gamma_0(11) \begin{pmatrix} 0 & -1 \\ 1 & d' \end{pmatrix}$ for $d \in \mathbb{Z}/11\mathbb{Z}$ and with d' denoting any integer from the residue class d, and $\infty \leftrightarrow \Gamma_0(11)$. For $P \in \mathbb{P}^1(\mathbb{Z}/11\mathbb{Z})$ set $\phi_P = \phi_{A,1,1}$ (A any matrix corresponding to P). Now a short calculation shows $\phi_0 = \phi_\infty = 0$, and for $d \neq 0, \infty$ the formula

$$C_{\phi P}(\Delta, r) = N_{\Delta,r}(d) - N_{\Delta,r}(-d) - \left(\left(\frac{rd^*}{11} \right) \right) - \left(\left(\frac{rd}{11} \right) \right) .$$

Here $N_{\Delta,r}(d)$ denotes the number of triples $(a, b, c) \in \mathbb{Z}^3$ which satisfy

$$b^2 - 4ac = \Delta,\ b^2 < \Delta,\ a > 0,\ a \equiv \frac{(b + r)d^*}{2} \bmod 11,\ c \equiv \frac{(b - r)d}{2} \bmod 11 ,$$

we use d^* for the inverse modulo 11 of d, and $((x))$ is the periodic function with period 1 which is 0 at the integers and equal to $x - \frac{1}{2}$ for $0 < x < 1$. In particular, one has $\phi_d = -\phi_{-d}$ for all d, and—as it is easily verified by calculating the first few coefficients of the ϕ_d—

$$\phi_1 = \frac{9}{11} T, \qquad \phi_2 = \frac{3}{11} T, \qquad \phi_3 = \phi_4, \qquad \phi_5 = \frac{-3}{11} T .$$

Computing the first few coefficients of $\mathscr{S}_{1,1}(\phi_3)$ shows that they coincide with those of $\frac{1}{5}(S + \frac{9}{11}E)$. From the theory quoted above we thus deduce $\mathscr{S}_{1,1}(5\phi_3 - \phi_1) = S$, which yields amusing formulas for the Fourier coefficients of

S, and that the values of $|\Delta|^{1/2}L(S, \Delta, 1)$ are proportional to the squares of the Fourier coefficients of $5\phi_3 - \phi_1$. The first few of these coefficients $C(\Delta, r)$ are

Δ	1	4	5	9	12	16	20	25	33	36	37	44	45	48	49	53	56	60	...
$C(\Delta, r_\Delta)$	1	−3	5	−2	5	4	5	0	0	6	5	0	0	10	−3	10	−10	−5	...

(For each Δ the symbol r_Δ denotes that solution of $r^2 \equiv \Delta \bmod 44$ which satisfies $0 \leqq r \leqq 11$).

The idea for the proof of the theorem is roughly as follows. Any elliptic cusp form on $\Gamma_0(m)$ of weight 2, considered as a holomorphic differential on the compactification of $\Gamma_0(m)\backslash\mathfrak{H}$, is determined by its periods. As paths in the upper half plane for computing the periods one may restrict to hyperbolic lines which connect 0 and rational numbers equivalent to 0 modulo $\Gamma_0(m)$. By the so-called 'Manin trick' these periods can be expressed as finite linear combinations of path integrals of the form $\int_{A0}^{Ai\infty}$ where A runs through a set of representatives of $\Gamma_0(m)\backslash SL_2(\mathbb{Z})$ (cf. [M]). Let $\mathscr{K}_{\Delta_0,r_0}(\tau, z; t)$ be the kernel function of the lifting map $\mathscr{S}_{\Delta_0,r_0}$ with respect to the Petersson scalar product. Since a linear combination of the $\mathscr{S}_{\Delta_0,r_0}$ is an injection it is clear that all the Jacobi forms $\langle \mathscr{K}_{\Delta_0,r_0}(\tau, z; \cdot) | f \rangle$ together generate $S^-_{2,m}$ and $S^+_{2,m}$ when f runs through the set of cusp forms on $\Gamma_0(m)$. By the above any such scalar product can be written as a linear combination of path integrals $\int_{A0}^{Ai\infty} \mathscr{K}_{\Delta_0,r_0}(\tau, z; t)\,dt$. Hence all these path integrals together generate $S^-_{2,m}$ and $S^+_{2,m}$. Thus, if we have explicit formulas for the kernel functions and if we can carry out explicitly the integration along paths joining $A0$ and $Ai\infty$, then we can prove a theorem like the one stated here. (Some of these sketched arguments are not literally true since certain Jacobi forms in $S^+_{2,m}$ lift to Eisenstein series so that, for instance, some of the above path integrals are not a priori defined.)

For the case of holomorphic Jacobi forms the kernels $\mathscr{K}_{\Delta_0,r_0}$ were constructed explicitly in [G-K-Z]. However, we do not take these kernels to deduce our theorem. Instead, we use certain non-holomorphic kernel functions. These are Jacobi theta series associated to quadratic forms of signature (2, 2). They represent the first non-trivial examples of Jacobi theta series which arise naturally when one mimics in the theory of Jacobi forms the well-known construction of the theta kernels which are used in the theory of dual reductive pairs (cf. [S2]).

There is a twofold reason for considering such theta kernels instead of the proper kernel functions $\mathscr{K}_{\Delta_0,r_0}$. First of all, the formula for $\mathscr{K}_{\Delta_0,r_0}$ as given in [G-K-Z] shows that an explicit computation of the integrals of $\mathscr{K}_{\Delta_0,r_0}$ along the paths from $A0$ and $Ai\infty$ involves some delicate convergence problems. In contrast to this the corresponding computations using the theta kernels turn out to be surprisingly simple. Secondly, we would like to treat the case of skew-holomorphic forms as well but corresponding results as for holomorphic Jacobi forms are not yet available in the literature. The theta kernels, apart from adding a new (admittedly not very surprising) aspect to the theory of Jacobi forms, allow us to give a self-contained proof of the theorem which includes holomorphic as well as skew-holomorphic Jacobi forms. Moreover, the investigation of these theta kernels as well as the computation of the corresponding path integrals exhibits, in our opinion, some amusing aspects.

There remain two points to be mentioned with respect to the proof. The minor one is that we shall not really consider the integrals $\int_{A0}^{Ai\infty}$ of the theta kernels but the symmetrization $\int_{A0}^{Ai\infty} + \int_{-A0}^{-Ai\infty}$ (for $\Delta_0 < 0$) or the antisymmetrization $\int_{A0}^{Ai\infty} - \int_{-A0}^{-Ai\infty}$ (for $\Delta_0 > 0$), both taken, so to speak, in the sense of the Cauchy principal value (to be precise, we consider $\int_0^{i\infty}([A]^* \pm [gAg]^*)$ where $[A]^*$ is the pullback operation on differentials of the isomorphism $[A]$ on the upper half plane induced by A and g denotes the diagonal matrix with -1 and 1 in the diagonal). The reason for this is that the first integrals would not exist in general since the theta kernels involve contributions which come from elliptic Eisenstein series. However, to consider these symmetrized or antisymmetrized versions makes perfect sense. The mappings which associate to an elliptic cusp form its path integrals $\int_{A0}^{Ai\infty} + \int_{-A0}^{-Ai\infty}$ or $\int_{A0}^{Ai\infty} - \int_{-A0}^{-Ai\infty}$ respectively are both injective. It can be shown that they define rational structures on the space of modular forms of weight 2 on $\Gamma_0(m)$. These rational structures are the natural generalizations to weight 2 forms on $\Gamma_0(m)$ of those rational structures considered in [K-Z], and they will be investigated (for arbitrary weight) in a forthcoming paper by J.A. Antoniadis.

A more serious point is that the proof of the theorem is not completely independent of the literature. We have to use the fact that the intersection of the kernels of all the $\mathscr{S}_{\Delta_0, r_0}$ equals 0, i.e. that for any nonzero Jacobi form there is at least one non-zero Fourier coefficient $C(\Delta_0 l^2, r_0 l)$ with some integer l and fundamental Δ_0 such that $r_0^2 \equiv \Delta_0 \bmod 4m$. This seems to be a fairly deep fact. Its proof depends on a trace formula for Jacobi forms and was given in [S-Z] for holomorphic Jacobi forms. A corresponding proof for the case of skew-holomorphic Jacobi forms is not yet available in the published literature and will be given in [S]. The suspicious reader may thus divide the stated theorem into two, one, unchanged but valid only for holomorphic Jacobi forms, and a second one for skew-holomorphic Jacobi forms, but which must then be stated in the weaker form that the span of the $\phi_{A, \Delta_0, r_0}(\tau, z)$ has as orthogonal complement in $S_{2,m}^+$ (with respect to the Petersson scalar product) the intersection of the kernels of the $\mathscr{S}_{\Delta_0, r_0}$. This is what we shall actually prove (cf. the end of section 2). It is perhaps worthwhile to mention that one could possibly circumvent this problem by considering more general theta kernels which would yield lifting maps $\mathscr{S}_{\Delta_0, r_0}$ associated to arbitrary (i.e. not necessarily fundamental) discriminants Δ_0. This would mean generalizing the Eq. (3) in Sect. 3, or, in other words, to study how many and which $SL_2(\mathbb{Z})$-invariant vectors are contained in the Weil representation associated to a certain finite quadratic module the definition of which can be read off from the cited identity. However, we did not pursue this here.

A theorem of similar type to the above was proved by Kohnen and Zagier in [K-Z]. They show that any modular form on $\Gamma_0(4)$, of weight $k + \frac{1}{2}$ (k even) from the Kohnen '$+$'-space, can be written as a linear combination of certain explicitly given functions which – as examples show – are in general not modular forms but look very much like theta series with spherical polynomials associated to the indefinite ternary form $b^2 - 4ac$. (In the quoted article this statement appears more as an incidental remark than as a theorem [loc. cit., p. 236]; moreover, their observation is strictly true only under the additional hypothesis that each Hecke eigenform of weight $k + \frac{1}{2}$ in Kohnen's '$+$'-space has non-zero first Fourier coefficient). Their proof is based on the fact that the coefficients of a Kohnen

'+'-space Hecke eigenform are essentially given by the periods of the (via Shimura lift) associated modular form of even weight around closed geodesics, and on an identity expressing such cycle integrals explicitly as linear combinations of the values at the integer points in the critical strip of the L-series of the associated form (loc. cit. Theorem 7). Our formulas for the Fourier coefficients of weight 2 modular forms on $\Gamma_0(m)$ should also be very closely related to the formulas given in [M] for the Fourier coefficients of Hecke eigenforms. It may be worthwhile to make this connection explicit.

Finally, it is possible to generalize our theorem to higher weight (cf. [S3]). The more general result amounts essentially to replacing the terms $\operatorname{sign}(Q \circ A)$ in the definition of $C_{A,\Delta_0,r_0}(\Delta, r)$ by $\operatorname{sign}(Q \circ A) \cdot P(a, b, c)$, where P is an arbitrary spherical polynomial (with respect to the quadratic form $b^2 - 4ac$), and the $\varepsilon_{m|\Delta_0|}(Q \circ A)$ by certain terms, which are derived from P and involve higher Bernoulli polynomials. However, there are certain non-trivial correction terms which have to be added in the case of weight strictly greater than 2. The precise result is too complicated to be stated here. Apart from these correction terms there is still one more point where the case of weight 2 treated here exhibits some special features compared to the higher weight case. In contrast to the usual expectation, the proof of the corresponding theorem for weight strictly greater than 2 does become harder. First of all, it cannot simply be carried out by mimicking the procedure of this paper (the appropriate period integrals of the theta kernels for higher weight do not lead automatically to holomorphic resp. skew-holomorphic Jacobi forms), and secondly, the computations given in [S3] (which use holomorphic or skew-holomorphic (i.e. proper) kernel functions instead of theta kernels) are partly somewhat painful. Because of some very subtle questions of convergence – as already indicated above – the computations in [S3] do not even comprise the weight 2 case as an obvious 'degenerate case'. For details the reader is referred to the paper loc. cit.

2. Proof

As in the theorem, fix a fundamental discriminant Δ_0 and an integer r_0 such that $\Delta_0 \equiv r_0^2 \bmod 4m$. Set

$$\Theta_{\Delta_0, r_0}(\tau, z; t) := \sum_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv r^2 \bmod 4m}} C_v(\Delta, r; t) e^{2\pi i\left(\frac{r^2 - \Delta}{4m} u + \frac{r^2 + \sigma\Delta}{4m} iv + rz\right)} \tag{1}$$

where

$$C_v(\Delta, r; t) = v^{\frac{1}{2}} \sum_{Q \in \mathcal{Q}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q) \frac{Q(t)}{\eta^2} \exp\left(-\frac{\pi v \hat{Q}(t)^2}{m|\Delta_0|\eta^2} \right)$$

and $\sigma = \operatorname{sign}(\Delta_0)$. Here

$$\tau = u + iv, \ t = \xi + i\eta \in \mathfrak{H}, \qquad z \in \mathbb{C}$$

($\mathfrak{H}$ = Poincaré upper half plane of complex numbers with positive imaginary parts) and we use the notation

$$\widehat{[a, b, c]}(t) := a|t|^2 + b\xi + c\,.$$

Taking absolute values and writing $[a, b, c]$ for Q and $(b^2 - 4ac)/\Delta_0$ for Δ we see that the series defining $\Theta_{\Delta_0, r_0}(\tau, z; t)$ is dominated by

$$v^{\frac{1}{2}} \sum_{r, a, b, c \in \mathbb{Z}} e^{-2\pi\left(\frac{r^2}{4m}v + r\mathrm{Im}(z)\right)} \frac{|at^2 + bt + c|}{\eta^2} e^{\frac{-\pi v}{2m|\Delta_0|}F_\eta(a, b, c)} \tag{2}$$

where $F_\eta(a, b, c) = (b^2 - 4ac) + \frac{2}{\eta^2}(a|t|^2 + b\xi + c)^2$. Since it is easily checked that, for fixed t, the quadratic form $F_\eta(a, b, c)$ is positive definite, we deduce that the series in (1) is normally convergent, i.e. uniformly convergent on compact subsets in the $(\tau, z; t)$-domain, and defines a smooth function in τ, z, t. Moreover $\Theta_{\Delta_0, r_0}(\tau, z; t)$ is obviously periodic in τ and z with period 1, and it can be checked by a tedious but routine application of the Poisson summation formula that $\Theta_{\Delta_0, r_0}(\tau, z; t)$ transforms like a Jacobi form of weight 2 and index m (holomorphic for $\Delta_0 < 0$, skew-holomorphic for $\Delta_0 > 0$) under $\tau \mapsto \frac{-1}{\tau}$, $z \mapsto \frac{z}{\tau}$. However, this also follows from the Proposition below and we skip this computation.

Fix a matrix $A \in SL_2(\mathbb{Z})$ and define

$$\phi(\tau, z) := \int_0^{i\infty} \left(\Theta_{\Delta_0, r_0}(\tau, z; At)\overline{\mathrm{d}(At)} - \mathrm{sign}(\Delta_0)\Theta_{\Delta_0, r_0}(\tau, z; A^*t)\overline{\mathrm{d}(A^*t)}\right). \tag{3}$$

Here $At = \frac{\alpha t + \beta}{\gamma t + \delta}$ and $A^* = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ for $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$; thus $\mathrm{d}(At) = \frac{\mathrm{d}t}{(\gamma\tau + \delta)^2}$ and $A^* = gAg$, $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The integral has to be taken along the line $t = i\eta$ with η ranging from 0 to ∞. We shall see below that the integral converges absolutely. Thus $\phi(\tau, z)$ is well defined and, of course, it inherits from $\Theta_{\Delta_0, r_0}(\tau, z; t)$ the periodicity in τ, z and the invariance property with respect to $\tau \mapsto \frac{-1}{\tau}$, $z \mapsto \frac{z}{\tau}$. Thus, it will be a Jacobi form if it has the correct Fourier development. In fact, we shall show next that $\phi(\tau, z)$ has a Fourier development which coincides, up to a non-zero constant, with that of $\phi_{A, \Delta_0, r_0}(\tau, z)$ as given in the theorem. This will then prove part of the theorem, namely that $\phi_{A, \Delta_0, r_0}(\tau, z)$ is a Jacobi form.

To compute $\phi(\tau, z)$ we first of all rewrite the integrand. Using the easily checked identities

$$\frac{Q(At)}{\mathrm{Im}(At)^2}(\gamma\bar{t} + \delta)^{-2} = \frac{Q \circ A(t)}{\mathrm{Im}(t)^2}, \qquad \frac{\hat{Q}(At)}{\mathrm{Im}(At)} = \frac{\widehat{Q \circ A}(t)}{\mathrm{Im}(t)} \tag{4}$$

we can write

$$\begin{aligned} \tilde{C}_v(\Delta, r; t) :=\ & C_v(\Delta, r; At)(\gamma\bar{t} + \delta)^{-2} - \mathrm{sign}(\Delta_0)C_v(\Delta, r; A^*t)(-\gamma\bar{t} + \delta)^{-2} \\ =\ & v^{\frac{1}{2}}\Bigg\{\sum_{Q \in \mathcal{Q}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q)\frac{Q \circ A(t)}{\eta^2} e^{-\lambda\frac{\widehat{Q \circ A}(t)^2}{\eta^2}} \\ & - \mathrm{sign}(\Delta_0) \sum_{Q \in \mathcal{Q}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q)\frac{Q \circ gAg(t)}{\eta^2} e^{-\lambda\frac{\widehat{Q \circ gAg}(t)^2}{\eta^2}}\Bigg\}, \end{aligned}$$

where we put $\lambda = \frac{\pi v}{m|\Delta_0|}$. Replace Q by $-Q \circ g$ in the second sum. Since $[a, b\, c] \circ g = [a, -b, c]$ we find that $\chi_{\Delta_0}(-Q \circ g) = \operatorname{sign}(\Delta_0)\chi_{\Delta_0}(Q)$ and that, for $t = i\eta$ and $Q \circ A =: [a, b, c]$, one has $Q \circ A(t) + Q \circ g^2 Ag(t) = 2(-a\eta^2 + c)$ and $\widehat{Q \circ g^2 Ag}(t) = \widehat{Q \circ A}(t) = a\eta^2 + c$. Thus

$$\tilde{C}_v(\Delta, r; i\eta) = \frac{2v^{\frac{1}{2}}}{\eta} \sum_{[a,b,c] \in \mathcal{Q}(\Delta\Delta_0, rr_0) \circ A} \chi_{\Delta_0}([a, b, c] \circ A^{-1})\left(-a\eta + \frac{c}{\eta}\right) e^{-\lambda\left(a\eta + \frac{c}{\eta}\right)^2}.$$

Take here now a typical term with $ac \neq 0$ and set $\eta = \sqrt{|\frac{c}{a}|}e^{\theta}$. Then

$$\int_0^\infty \left(-a\eta + \frac{c}{\eta}\right) e^{-\lambda\left(a\eta + \frac{c}{\eta}\right)^2} \frac{d\eta}{\eta} = -\operatorname{sign}(a)\sqrt{|ac|} \int_{-\infty}^{+\infty} e^{-\lambda|ac|c(\theta)^2} dc(\theta),$$

where $c(\theta) = 2\cosh(\theta)$ if $ac > 0$ and $c(\theta) = 2\sinh(\theta)$ if $ac < 0$. Thus the latter integral vanishes for $ac > 0$ (since then the integrand is odd) and otherwise equals $-\operatorname{sign}(a)\sqrt{|ac|}\int_{-\infty}^{+\infty} e^{-\lambda|ac|x^2} dx = -\operatorname{sign}(a)\sqrt{\frac{m|\Delta_0|}{v}}$.

To handle terms with $ac = 0$ we rewrite the contribution to $\tilde{C}_v(\Delta, r; t)$ resulting from these terms as

$$-2v^{\frac{1}{2}} \left\{ \sum_{\substack{a \bmod m|\Delta_0|,\, b \in \mathbb{Z} \\ [a,b,0] \in \mathcal{Q}(\Delta\Delta_0, rr_0) \circ A}} \chi_{\Delta_0}([a, b, 0] \circ A^{-1})\theta_a(\eta) - \sum_{\substack{c \bmod m|\Delta_0|,\, b \in \mathbb{Z} \\ [0,b,c] \in \mathcal{Q}(\Delta\Delta_0, rr_0) \circ A}} \chi_{\Delta_0}([0, b, c] \circ A^{-1}) \frac{\theta_c(1/\eta)}{\eta^2} \right\}$$

where

$$\theta_a(\eta) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv a \bmod m|\Delta_0|}} x e^{-\lambda x^2 \eta^2}$$

and where we used that $\chi_{\Delta_0}([a, b, c] \circ A^{-1})$ depends on a only modulo $m|\Delta_0|$. By a standard computation we find

$$\int_0^\infty \theta_a(\eta) d\eta = \frac{1}{2}\sqrt{\frac{m|\Delta_0|}{v}} \left(\zeta\left(0, \frac{a}{m|\Delta_0|}\right) - \zeta\left(0, \frac{-a}{m|\Delta_0|}\right)\right),$$

where

$$\zeta(0, u) = \sum_{\substack{x > 0 \\ x \equiv u \bmod \mathbb{Z}}} \frac{1}{x^s}\bigg|_{s=0} = \frac{1}{2} - u \qquad \text{for } 0 < u \leqq 1.$$

The integral $\int_0^\infty \theta_c\left(\frac{1}{\eta}\right)\frac{1}{\eta^2} d\eta$ is treated in exactly the same way after substituting $\eta \mapsto \frac{1}{\eta}$.

Summarizing we thus have

$$\int_0^\infty \tilde{C}_v(\Delta, r; i\eta) \overline{d(i\eta)} = 2i\sqrt{m|\Delta_0|} \times (\Delta, r\text{-th coefficient of } \phi_{A, \Delta_0, r_0}(\tau, z))$$

– up to a justification for the necessary interchange of summation and integration, which will be given below. But then, replacing the coefficients $\overline{C_v(\Delta, r; t)}$ in the defining equation (1) of $\Theta_{\Delta_0, r_0}(\tau, z; t)$ by the integrals $\int_0^\infty \tilde{C}_v(\Delta, r; \eta)\overline{\mathrm{d}(i\eta)}$ in order to obtain $\phi(\tau, z)$, we deduce – again the justification for the interchange of summation and integration is postponed for the moment, – that

$$\phi(\tau, z) = 2i\sqrt{m|\Delta_0|}\,\phi_{A, \Delta_0, r_0}(\tau, z)\,.$$

When doing the latter replacement note that one can at the same time replace the $\sigma\Delta$ in (1) by $|\Delta|$ since the integral $\int_0^\infty \tilde{C}_v(\Delta, r; \eta)\overline{\mathrm{d}(i\eta)}$ vanishes for $\Delta\Delta_0 < 0$.

It may be worthwhile to note that ϕ_{A, Δ_0, r_0} is in general not a cusp form, whereas for square free m it always has to be (since then there exist no non-cusp forms of weight 2), i.e. for square free m and any r with $r^2 \equiv 0 \bmod 4m$ one has

$$\sum_{\substack{0 < a < m|\Delta_0| \\ [a,0,0] \in \mathscr{Q}(0, rr_0) \circ A}} \chi_{\Delta_0}([a, 0, 0] \circ A^{-1})\left(\frac{a}{m|\Delta_0|} - \frac{1}{2}\right)$$

$$= \sum_{\substack{0 < c < m|\Delta_0| \\ [0,0,c] \in \mathscr{Q}(0, rr_0) \circ A}} \chi_{\Delta_0}([0, 0, c] \circ A^{-1})\left(\frac{c}{m|\Delta_0|} - \frac{1}{2}\right).$$

This can also be checked by a standard computation. (In fact, by the Dirichlet class number formula each side equals $-h'(\Delta_0)$ if $\Delta_0 < 0$ and $rr_0 \equiv 0 \bmod 2m$, and 0 otherwise, where $h'(\Delta_0)$ denotes the class number of $\mathbb{Q}(\sqrt{\Delta_0})$ for $\Delta_0 < -4$, and $h'(-3) = \frac{1}{3}$, $h'(-4) = \frac{1}{2}$.)

This completes the proof that $\phi_{A, \Delta_0, r_0}(\tau, z)$ is a Jacobi form apart from some estimates to justify the above interchanges of summation and integration. So assume first of all that $ac \neq 0$. Then one has

$$\mathrm{e}^{2\lambda ac} \int_0^\infty \left| -a\eta + \frac{c}{\eta} \right| \mathrm{e}^{-\lambda\left(a\eta + \frac{c}{\eta}\right)^2} \frac{\mathrm{d}\eta}{\eta} = \left(\int_0^{\sqrt{|c/a|}} + \int_{\sqrt{|c/a|}}^\infty \right) \left| -a\eta + \frac{c}{\eta} \right| \mathrm{e}^{-\lambda\left(a^2\eta^2 + \frac{c^2}{\eta^2}\right)} \frac{\mathrm{d}\eta}{\eta}$$

$$= 2 \int_{\sqrt{|c/a|}}^\infty (\text{“same”}) \quad (\text{replace } \eta \text{ by } |c/a|\eta^{-1})$$

$$\leqq 2 \int_{\sqrt{|c/a|}}^\infty |2a\eta| \mathrm{e}^{-\lambda a^2\eta^2} \frac{\mathrm{d}\eta}{\sqrt{|c/a|}}$$

$$= \frac{2}{\lambda\sqrt{|ac|}} \mathrm{e}^{-\lambda|ac|}\,.$$

Furthermore

$$\int_0^\infty |\theta_a(\eta)|\,\mathrm{d}\eta = \lambda^{-\frac{1}{2}} \int_0^\infty \left| \sum_{\substack{x \in \mathbb{Z} \\ x \equiv a \bmod m|\Delta_0|}} x\mathrm{e}^{-\eta^2 x^2} \right| \mathrm{d}\eta$$

(replace η by $\lambda^{\frac{1}{2}}\eta$ in the first integral), and the latter integral is bounded by a constant γ independent of a. Now

$$\int_0^{i\infty} \left| \Theta_{\Delta_0, r_0}(\tau, z; At)\overline{\mathrm{d}(At)} - \operatorname{sign}(\Delta_0)\Theta_{\Delta_0, r_0}(\tau, z; A^*t)\overline{\mathrm{d}(A^*t)} \right|$$

$$\leqq 2v^{\frac{1}{2}} \sum_{r \in \mathbb{Z}} \mathrm{e}^{-2\pi\left(\frac{r^2}{4m}v + r\operatorname{Im}(z)\right)} \left\{ \sum_{\substack{a, b, c \in \mathbb{Z} \\ ac \neq 0}} \mathrm{e}^{-\pi \frac{(b^2-4ac)}{2m|\Delta_0|}v} \int_0^{\infty} \left| -a\eta + \frac{c}{\eta} \right| \mathrm{e}^{-\frac{\pi v}{m|\Delta_0|}\left(a\eta + \frac{c}{\eta}\right)^2} \frac{\mathrm{d}\eta}{\eta} \right.$$

$$\left. + \sum_{a, c \bmod m|\Delta_0|} \left(\int_0^{\infty} |\theta_a(\eta)| \mathrm{d}\eta + \int_0^{\infty} |\theta_c(\eta)| \mathrm{d}\eta \right) \right\}$$

where we wrote $(b^2 - 4ac)/\Delta_0$ for Δ in the definition of $\Theta_{\Delta_0, r_0}(\tau, z; t)$ and where we have eventually enlarged by summing over all $a, b, c \in \mathbb{Z}$. By the given estimates for the integrals the right hand side is clearly convergent. Thus, we can apply Lebesgue's theorem to justify the interchanges of sums and integrals in the given computation of $\phi_{A, \Delta_0, r_0}(\tau, z)$. Note that we have also proved

$$\int_0^{i\infty} | \Theta_{\Delta_0, r_0}(\tau, z; At)\overline{\mathrm{d}(At)} - \operatorname{sign}(\Delta_0)\Theta_{\Delta_0, r_0}(\tau, z; A^*t)\overline{\mathrm{d}(A^*t)} | \mathrm{e}^{-2\pi m \operatorname{Im}(z)^2/v} = \mathcal{O}(1) \tag{5}$$

for $v \to \infty$ where the $\mathcal{O}$-constant is independent of u and z.

To prove the second part of the theorem, i.e. that the cuspidal parts of the $\phi_{A, \Delta_0, r_0}(\tau, z)$ span $S^-_{2,m}$ and $S^+_{2,m}$ we have to introduce the Petersson scalar product. Let $\phi(\tau, z)$ be any Jacobi cusp form from $S^-_{2,m}$ or $S^+_{2,m}$, and let $\psi(\tau, z)$ be any – say smooth – function on $\mathfrak{H} \times \mathbb{C}$ such that $\psi(\tau, z)\mathrm{e}^{-2\pi m \operatorname{Im}(z)^2/v} = \mathcal{O}(v^k)$ for $v \to \infty$ with some k and an $\mathcal{O}$-constant which is independent of u and z. Then the Petersson scalar product $\langle \phi | \psi \rangle$ of $\phi(\tau, z)$ and $\psi(\tau, z)$ is defined by

$$\langle \phi | \psi \rangle = \frac{1}{2} \int_{\mathfrak{F}} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \phi(\tau, \lambda\tau + \mu)\overline{\psi(\tau, \lambda\tau + \mu)} \mathrm{e}^{-4\pi m \lambda^2 v} \mathrm{d}\lambda \mathrm{d}\mu \mathrm{d}u \mathrm{d}v \,.$$

Here the integral has to be taken with respect to u and v over the standard fundamental domain $\mathfrak{F}$ for $\mathfrak{H}$ modulo $SL_2(\mathbb{Z})$, i.e. $\mathfrak{F} = \{\tau \in \mathfrak{H} | -\frac{1}{2} \leqq u \leqq \frac{1}{2}, |\tau| \geqq 1\}$. The integral is absolutely convergent since for a cusp form $\phi(\tau, z)$ the expression $\phi(\tau, z)\mathrm{e}^{-2\pi m \operatorname{Im}(z)^2/v}$ is exponentially decreasing uniformly in u and z for $v \to \infty$. In particular, it defines a non-degenerate scalar product on the finite dimensional spaces $S^-_{2,m}$ and $S^+_{2,m}$. Note that the λ, μ-domain of integration is invariant under $(\lambda, \mu) \mapsto (-\lambda, -\mu)$. Thus, $\langle \phi | \psi \rangle = 0$ whenever $\phi(\tau, z)\overline{\psi(\tau, z)}$ defines an odd function with respect to the variable z. The latter holds true e.g. in the case that ϕ is a holomorphic (or skew-holomorphic) cusp form and ψ satisfies the same transformation law under $\tau, z \mapsto \frac{-1}{\tau}, \frac{z}{\tau}$ as a skew-holomorphic (or holomorphic) cusp form; namely, applying twice this transformation law to such a $\psi(\tau, z)$ shows that $\psi(\tau, z)$ is an even function in z in the holomorphic, and an odd function in z in the skew-holomorphic case.

In the next section we shall need a more conceptual way to look at or the Petersson scalar product. To explain this denote by $\mathcal{J}(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ the Jacobi-group over $\mathbb{Z}$, i.e. the set of all pairs $A[\lambda, \mu]$ ($A \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$)

equipped with the multiplication

$$A[\lambda,\mu]\cdot A'[\lambda',\mu'] = AA'[(\lambda,\mu)A' + (\lambda',\mu')] .$$

The Jacobi group acts on $\mathfrak{H}\times\mathbb{C}$ by $\Upsilon\cdot(\tau,z) = \left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z+\lambda\tau+\mu}{\gamma\tau+\delta}\right)$ (for $\Upsilon = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ $[\lambda,\mu]$) and on functions $\phi(\tau,z)$ by

$$\left(\phi|^-\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)(\tau,z) = \frac{e^{-2\pi i m\frac{\gamma z^2}{\gamma\tau+\delta}}}{(\gamma\tau+\delta)^2}\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta}\right)$$

and

$$\phi|^-[\lambda,\mu](\tau,z) = e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) ,$$

and also by

$$\left(\phi|^+\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)(\tau,z) = \frac{e^{-2\pi i m\frac{\gamma z^2}{\gamma\tau+\delta}}}{(\gamma\bar\tau+\delta)|\gamma\tau+\delta|}\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta}\right)$$

and $\phi|^+[\lambda,\mu] = \phi|^-[\lambda,\mu]$. It is easily checked that a holomorphic or skew-holomorphic Jacobi form ϕ (of weight 2 and index m) satisfies $\phi|^-\Upsilon = \phi$ or $\phi|^+\Upsilon = \phi$, respectively, for all $\Upsilon\in\mathcal{J}(\mathbb{Z})$. If $\phi(\tau,z)$ and $\psi(\tau,z)$ are as above and if $\psi(\tau,z)$ additionally satisfies the same transformation law with respect to $\mathcal{J}(\mathbb{Z})$ as $\phi(\tau,z)$ then the expression $\phi(\tau,z)\overline{\psi(\tau,z)}e^{-4\pi m y^2/v}v^2$ $(z = x+iy)$ is invariant by replacing (τ,z) by $\Upsilon\cdot(\tau,z)$ for all $\Upsilon\in\mathcal{J}(\mathbb{Z})$. Moreover

$$\langle\phi|\psi\rangle = \int_{\mathcal{J}(\mathbb{Z})\backslash\mathfrak{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}e^{-4\pi m y^2/v}v^2\,dV ,$$

where $dV = \frac{du\,dv\,dx\,dy}{v^3}$ is the $\mathcal{J}(\mathbb{Z})$-invariant volume element on $\mathfrak{H}\times\mathbb{C}$ and the integral has to be taken over any fundamental domain of $\mathfrak{H}\times\mathbb{C}$ modulo $\mathcal{J}(\mathbb{Z})$. (For more details or proofs of the facts listed in this paragraph the reader is referred to [E-Z].)

Let now $\phi(\tau,z)$ in $S^-_{2,m}$ and $S^+_{2,m}$ be from the orthogonal complement of the span of the $\phi_{A,\Delta_0,r_0}(\tau,z)$. Thus

$$\langle\phi|\phi_{A,\Delta_0,r_0}\rangle = 0 \tag{6}$$

for all A, Δ_0, r_0. We have to prove that (6) implies $\phi\equiv 0$. To interpret (6) note that $\overline{\Theta_{\Delta_0,r_0}(\tau,z;t)}$ as a function of t, for fixed τ and z behaves like an element from $M_2(\Gamma_0(m))$, the space of elliptic modular forms on $\Gamma_0(m)$ of weight 2. Indeed, the transformation law $\overline{\Theta_{\Delta_0,r_0}\left(\tau,z;\frac{\alpha t+\beta}{\gamma t+\delta}\right)}(\gamma\bar t+\delta)^{-2} = \overline{\Theta_{\Delta_0,r_0}(\tau,z;t)}$ for all $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(m)$ is immediately clear from the definition of $\Theta_{\Delta_0,r_0}(\tau,z;t)$ using (4) and the invariance of χ_{Δ_0} and the $\mathcal{Q}(\Delta,r)$ under $\Gamma_0(m)$. This transformation law with respect to $\Gamma_0(m)$ is then also fulfilled by $f(t) := \langle\phi|\Theta_{\Delta_0,r_0}(\cdot,\cdot;t)\rangle$. Here, of course, we have to check that the Petersson scalar product is defined, i.e. that $\Theta_{\Delta_0,r_0}(\tau,z;t)$

as a function in τ, z satisfies the correct boundedness condition. But this follows easily from (2), which moreover shows that $\Theta_{\Delta_0, r_0}(\tau, z; t)\mathrm{e}^{-2\pi m y^2/v}$ can be bounded by a polynomial in v and η which does not depend on u, z, ξ. Thus, for any cusp form ϕ, the function $f(t)$ is smooth on $\mathfrak{H}$ and can be bounded by a polynomial in η independently of ξ. In fact, we shall show below that $f(t)$ is an element of $M_2(\Gamma_0(m))$. Using (5) to justify the interchange of integrals we may rewrite (6) as

$$\int_0^{i\infty} (\langle \phi | \Theta_{\Delta_0, r_0}(\cdot, \cdot; At)\rangle \mathrm{d}(At) - \operatorname{sign}(\Delta_0)\langle \phi | \Theta_{\Delta_0, r_0}(\cdot, \cdot; A^*t)\rangle \mathrm{d}(A^*t)) = 0 \quad (7)$$

and this is then a statement about modular forms. We apply the following Lemma.

Lemma. *Let $f(t)$ be a modular form from $M_2(\Gamma_0(m))$, let $\varepsilon \in \{\pm 1\}$. Assume that for each $A \in SL_2(\mathbb{Z})$ the integral*

$$\int_0^{i\infty} (f(At)\mathrm{d}(At) + \varepsilon f(A^*t)\mathrm{d}(A^*t)),$$

taken along the path $t = i\eta$, $\eta > 0$, is absolutely convergent and equal to 0. Then $f(t)$ is identically 0.

This Lemma is probably known to the specialists, but for the sake of completeness we give the short proof in Sect. 4.

Applying this Lemma we deduce from (7) that $\langle \phi | \Theta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle = 0$ for all Δ_0, r_0. To investigate this we need the following Proposition.

Proposition. *For $\tau, t \in \mathfrak{H}$, $z \in \mathbb{C}$, $\tau = u + iv$, $t = \xi + i\eta$ define*

$$\theta(\tau, z, t) = \frac{\partial}{\partial t} \sum_{\substack{r, s \in \mathbb{Z} \\ r \equiv s r_0 \bmod 2m}} \mathrm{e}^{2\pi i\left(\frac{r^2 - s^2\Delta_0}{4m}u + \frac{r^2 + s^2|\Delta_0|}{4m}iv + rz + s|\Delta_0|\xi\right)} \mathrm{e}^{-\pi m|\Delta_0|\eta^2/v}.$$

Then one has

$$\Theta_{\Delta_0, r_0}(\tau, z; t)$$

$$= \frac{\sqrt{m}}{\pi\varepsilon}\left\{\left(\frac{\Delta_0}{0}\right)\theta(\tau, z, 0) + \sum_A \sum_{l \geqq 1} \frac{|\Delta_0|}{l}\left(\frac{\Delta_0}{l}\right)\frac{\mathrm{e}^{-2\pi i m\frac{\gamma z^2}{\gamma\tau + \delta}}}{(\gamma\tau + \delta)^{\times}}\theta\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \frac{z}{\gamma\tau + \delta}, \frac{lt}{\Delta_0}\right)\right\}.$$

Here A runs through a set of representatives $A = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ for $\begin{pmatrix}1 & \mathbb{Z} \\ 0 & 1\end{pmatrix}\backslash SL_2(\mathbb{Z})$ and for each such A the expression $(\gamma\tau + \delta)^{\times}$ equals $(\gamma\tau + \delta)^2$ for negative Δ_0, and it equals $(\gamma\bar{\tau} + \delta)|\gamma\tau + \delta|$ for positive Δ_0. Moreover, $\varepsilon = \sqrt{\operatorname{sign}(\Delta_0)}$ (where $\sqrt{-1} = i$), and $\left(\frac{\Delta_0}{0}\right) = 1$ if $\Delta_0 = 1$ and $\left(\frac{\Delta_0}{0}\right) = 0$ otherwise.

We shall prove the proposition in the next section. Note that the given formula for $\Theta_{\Delta_0, r_0}(\tau, z; t)$ may also be written as

$$\Theta_{\Delta_0, r_0}(\tau, z; t) = \frac{\sqrt{m}|\Delta_0|}{\pi\varepsilon}\left\{\left(\frac{\Delta_0}{0}\right)\pi i\, T_{r_0}(\tau, z) + \sum_{\Upsilon} \sum_{l \geqq 1} \sum_{s \in \mathbb{Z}} \frac{1}{l}\left(\frac{\Delta_0}{l}\right)\frac{\partial}{\partial t}\left(\kappa_{\Delta_0 s^2, r_0 s}|^{\pm}\Upsilon\right)(\tau, z; lst)\right\}$$

with

$$\kappa_{\Delta,r}(\tau, z; t) = e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}iv + rz\right)} e^{2\pi i \operatorname{sign}(\Delta)\xi} e^{-\frac{\pi m\eta^2}{|\Delta|v}}$$

and

$$T_{r_0}(\tau, z) = \sum_{\substack{r, s \in \mathbb{Z} \\ r \equiv sr_0 \bmod 2m}} s e^{2\pi i\left(\frac{r^2-s^2}{4m}u + \frac{r^2+s^2}{4m}iv + rz\right)}.$$

Here Υ runs through a complete set of representatives for $\mathcal{J}(\mathbb{Z})_\infty \backslash \mathcal{J}(\mathbb{Z})$ (with $\mathcal{J}(\mathbb{Z})_\infty = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}[0, \mathbb{Z}]$), the $\pm$ in $|^{\pm}$ is determined by $\pm 1 = \operatorname{sign}(\Delta_0)$, and the action $|^{\pm}$ refers to the first pair of variables, of course. A simple consequence of this formula is then

Corollary. *Let $\phi(\tau, z)$ be a Jacobi cusp form from $S_{2,m}^-$ or $S_{2,m}^+$ with Fourier coefficients $C_\phi(\Delta, r)$. Then $\langle \phi \,|\, \Theta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle$ is a modular form of weight 2 on $\Gamma_0(m)$ with Fourier expansion*

$$\frac{\langle \phi \,|\, \Theta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle}{(-i\bar{\varepsilon})\sqrt{m|\Delta_0|}} = \left(\frac{\Delta_0}{0}\right)\langle \phi \,|\, T_{r_0}\rangle + \sum_{l \geqq 1}\left(\sum_{a|l}\left(\frac{\Delta_0}{a}\right) C_\phi\left(\Delta_0 \frac{l^2}{a^2}, r_0 \frac{l}{a}\right)\right) e^{2\pi i l t}$$

From the Corollary (proof in the next section) we now obtain that our $\phi(\tau, z)$, i.e. any $\phi(\tau, z)$ which is orthogonal to the span of the $\Theta_{\Delta_0, r_0}(\tau, z; t)$, must necessarily satisfy $C_\phi(\Delta_0 l^2, r_0 l) = 0$ for all $l \geqq 1$ and all fundamental discriminants Δ_0 and all r_0 such that $\Delta_0 \equiv r_0^2 \bmod 4m$, or, equivalently, that it must necessarily lie in the intersection of the kernels of all the lifting maps $\phi(\tau, z) \mapsto \langle \phi \,|\, \Theta_{\Delta_0, r_0}(\cdot, \cdot; t)\rangle$. But this implies $\phi \equiv 0$. For holomorphic Jacobi forms this was proved in [S-Z] (Theorem 3), for skew-holomorphic ones this will be proved in [S]. This completes the proof of the theorem.

3. Proof of the proposition and its corollary

Summing over $Q = [a, b, c]$ and replacing the discriminants Δ by $(b^2 - 4mac)/\Delta_0$ in the definition of $\Theta_{\Delta_0, r_0}(\tau, z; t)$ we can write

$$\Theta_{\Delta_0, r_0}(\tau, z; t) = \frac{v^{\frac{1}{2}}}{\eta^2} \sum_{\substack{r, a, b, c \in \mathbb{Z} \\ b \equiv rr_0 \bmod 2m}} e^{2\pi i\left(\frac{r^2\Delta_0 - b^2}{4m\Delta_0}u + \frac{r^2|\Delta_0| + b^2}{4m|\Delta_0|}iv + rz\right)} f_{\tau, t}(r, a, b)$$

where

$$f_{\tau, t}(r, a, b) = \sum_{\substack{c \in \mathbb{Z} \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} e^{2\pi i\left(\frac{ac}{\Delta_0}u - \frac{ac}{|\Delta_0|}iv\right)} \times$$

$$\times\ \chi_{\Delta_0}([ma, b, c])(mat^2 + bt + c) e^{-\frac{\pi v}{m|\Delta_0|\eta^2}(am|t|^2 + b\xi + c)^2}.$$

To $f_{\tau,t}(r, a, b)$, the sum over c, we now apply the Poisson summation formula. Thus we write

$$f_{r,t}(r, a, b) = \sum_{d \in \mathbb{Z}} \psi_{r,a,b}(d) g\left(\frac{d}{|\Delta_0|}\right)$$

where

$$\psi_{r,a,b}(d) = \frac{1}{|\Delta_0|} \sum_{\substack{c \bmod |\Delta_0| \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} \chi_{\Delta_0}([ma, b, c]) e^{2\pi i \frac{cd}{|\Delta_0|}}$$

and

$$g(d) = \int_{-\infty}^{+\infty} e^{-2\pi i \frac{a\tilde{\tau}}{|\Delta_0|} c} (mat^2 + bt + c) e^{-\frac{\pi v}{m|\Delta_0|\eta^2}(ma|t|^2 + b\xi + c)^2} e^{-2\pi i c d} \mathrm{d}c \,.$$

Here we used that the value $\chi_{\Delta_0}([ma, b, c])$, for fixed a, b, depends on c only modulo $|\Delta_0|$. Furthermore we use

$$\tilde{\tau} = \begin{cases} \tau & \text{if } \Delta_0 < 0 \\ -\bar{\tau} & \text{if } \Delta_0 > 0 \end{cases}.$$

Now, by a simple computation

$$g(d) = \frac{\sqrt{m|\Delta_0|}\,\eta^2}{\pi v^{\frac{1}{2}}} \frac{1}{\frac{a}{|\Delta_0|}\tilde{\tau} + d} \frac{\partial}{\partial \bar{t}} e^{2\pi i (ma\xi^2 + b\xi)\left(\frac{a}{|\Delta_0|}\tilde{\tau} + d\right)} e^{-\pi m|\Delta_0|\eta^2 \frac{\left|\frac{a}{|\Delta_0|}\tilde{\tau} + d\right|^2}{v}}$$

for $(a, d) \neq 0$, and

$$g(d) = i\left(\frac{m|\Delta_0|}{v}\right)^{\frac{1}{2}} \eta^2 b$$

for $(a, d) = 0$. For $(a, d) \neq 0$ let l be the greatest common divisor of a, d, set $\gamma = a/l$, $\delta = d/l$ and choose any matrix A in $SL_2(\mathbb{Z})$ such that $A = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$. Then we can write

$$g\left(\frac{l\delta}{|\Delta_0|}\right) = \frac{\sqrt{m}|\Delta_0|^{\frac{3}{2}}\eta^2}{\pi v^{\frac{1}{2}}} \frac{1}{l(\gamma\tilde{\tau} + \delta)} \frac{\partial}{\partial \bar{t}} e^{2\pi i (ml\gamma\xi^2 + b\xi)\frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}} e^{-\pi \frac{ml^2\eta^2}{|\Delta_0| \mathrm{Im}(A\tilde{\tau})}} \,.$$

Inserting this into the formula for $f_{\tau,t}(r, a, b)$ and then summing in the resulting formula for $\Theta_{\Delta_0, r_0}(\tau, z; t)$ over $l \geqq 1$ and a complete set of representatives A for $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash SL_2(\mathbb{Z})$ instead of $a, d \in \mathbb{Z}$ we obtain

$$\Theta_{\Delta_0, r_0}(\tau, z; t) = \frac{\sqrt{m|\Delta_0|}}{2\pi} \frac{\partial}{\partial z'} \varphi(\tau, z, 0; 0, 0)$$

$$+ \frac{\sqrt{m}|\Delta_0|^{\frac{3}{2}}}{\pi} \sum_A \sum_{l \geqq 1} \frac{1}{l(\gamma\tilde{\tau} + \delta)} \frac{\partial}{\partial \bar{t}} e^{2\pi i m l \gamma \xi^2 \frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}} \varphi\left(\tau, z, \xi \frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}; l\gamma, l\delta\right) e^{-\pi \frac{ml^2\eta^2}{|\Delta_0| \mathrm{Im}(A\tilde{\tau})}} \,.$$

Here for any $a, d \in \mathbb{Z}$ and any $z' \in \mathbb{C}$

$$\varphi(\tau, z, z'; a, d) = \sum_{\substack{r, b \in \mathbb{Z} \\ b \equiv r r_0 \bmod 2m}} \psi_{r, a, b}(d) e^{2\pi i\left(\frac{r^2 \Delta_0 - b^2}{4m\Delta_0} u + \frac{r^2|\Delta_0| + b^2}{4m|\Delta_0|} iv + rz + bz'\right)} .$$

We shall prove in a moment that

$$e^{2\pi i m l \gamma \xi^2 \frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}} \varphi\left(\tau, z, \xi \frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}; l\gamma, l\delta\right)$$
$$= \frac{e^{-2\pi i m \frac{\gamma' z^2}{\gamma'\tau + \delta'}}}{[(\gamma'\tau + \delta')(\gamma\tilde{\tau} + \delta)]^{\frac{1}{2}}} \varphi\left(A'\tau, \frac{z}{(\gamma'\tau + \delta')}, \frac{l\xi}{|\Delta_0|}; 0, l\right) . \tag{1}$$

Here on the right hand side that squareroot has to be taken which is positive or has positive imaginary part. Moreover $A' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ equals A if Δ_0 is negative and it equals $\begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ if Δ_0 is positive. Now

$$\psi_{r, 0, b}(l) = \frac{1}{|\Delta_0|} \sum_{c \bmod |\Delta_0|} \chi_{\Delta_0}([0, b, c]) e^{2\pi i \frac{cl}{|\Delta_0|}}$$

if $r^2 \Delta_0 \equiv b^2 \bmod 4m|\Delta_0|$ and $= 0$ otherwise. So assume $r^2 \Delta_0 \equiv b^2 \bmod 4m|\Delta_0|$. Then $\Delta_0 | b^2$, thus $\chi_{\Delta_0}([0, b, c]) = \left(\frac{\Delta_0}{c}\right)$ and hence $\psi_{r, 0, b}(l) = \left(\frac{\Delta_0}{l}\right) \varepsilon |\Delta_0|^{-\frac{1}{2}}$ since Δ_0 is fundamental (recall $\varepsilon = i$, 1 for $\Delta_0 < 0$, > 0, respectively). For the same reason we find that $r^2 \equiv \frac{b^2}{|\Delta_0|} \bmod 4m$ and $b \equiv r r_0 \bmod 2m$ imply $\Delta_0 | b$ and $r \equiv \frac{b}{\Delta_0} r_0 \bmod 2m$, and vice versa. Thus, summing in the sum defining $\varphi(\tau, z, \xi; 0, l)$ over $\Delta_0 s$ instead of b we can write in the notation of the proposition

$$\frac{\partial}{\partial t} \varphi(\tau, z, \operatorname{sign}(\Delta_0)\xi; 0, l) e^{-\pi m |\Delta_0| \eta^2/v} = \left(\frac{\Delta_0}{l}\right) \varepsilon |\Delta_0|^{-\frac{1}{2}} \theta(\tau, z; t) .$$

Inserting this in the last formula for $\Theta_{\Delta_0, r_0}(\tau, z; t)$ and summing over A' instead of A if $\Delta_0 > 0$, thereby noticing that $A(\tilde{\tau}) = A(-\bar{\tau}) = -A'\bar{\tau}$, we now easily recognize the asserted formula.

To prove (1) we write first of all for $a, d \in \mathbb{Z}$

$$\varphi(\tau, z, z'; a, d) = \sum_{\substack{r \bmod 2m,\ b \bmod 2m|\Delta_0| \\ b \equiv r r_0 \bmod 2m}} \psi_{r, a, b}(d) \vartheta_{m, r}(\tau, z) \vartheta_{m|\Delta_0|, b}(\tilde{\tau}, z') ,$$

where $\vartheta_{N, \rho}$ for any N, ρ is the basic function

$$\vartheta_{N, \rho}(\tau, z) = \sum_{\substack{s \in \mathbb{Z} \\ s \equiv \rho \bmod 2N}} e^{2\pi i\left(\frac{s^2}{4N}\tau + sz\right)} .$$

We shall show that for all $a, d \in \mathbb{Z}$ and for all $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$

$$\frac{e^{-2\pi i m\left(\frac{\gamma' z^2}{\gamma'\tau + \delta'} + |\Delta_0| \frac{\gamma z'^2}{\gamma\tilde{\tau} + \delta}\right)}}{[(\gamma'\tau + \delta')(\gamma\tilde{\tau} + \delta)]^{\frac{1}{2}}} \cdot \varphi\left(\frac{\alpha'\tau + \beta'}{\gamma'\tau + \delta'}, \frac{z}{\gamma'\tau + \delta'}, \frac{z'}{\gamma\tilde{\tau} + \delta}; (a, d)A^{-1}\right) = \varphi(\tau, z, z'; a, d) \tag{2}$$

where $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = A$ if $\Delta_0 < 0$ and $= \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ otherwise. Replacing z' by $\xi \frac{l(\gamma\tilde{\tau} + \delta)}{|\Delta_0|}$ and a, d by $l\gamma, l\delta$ this then clearly implies (1).

Now the left hand side of (2) defines an action of $SL_2(\mathbb{Z})$ on functions in τ, z, z' as is easily proved. Thus to verify (2) it suffices to check it for some generators A of $SL_2(\mathbb{Z})$ – say $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The first case is trivially verified. To treat the second one recall (or prove by Poisson summation)

$$\frac{e^{-2\pi i N \frac{z^2}{\tau}}}{\tau^{\frac{1}{2}}} \vartheta_{N,\rho}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2N}} \sum_{\sigma \bmod 2N} e^{\frac{-2\pi i \rho\sigma}{2N}} \vartheta_{N,\sigma}(\tau, z) .$$

Thus, for $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the left hand side of (2) becomes

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{1}{2}}} \sum_{\substack{r \bmod 2m \\ b \bmod 2m|\Delta_0| \\ b \equiv rr_0 \bmod 2m}} \psi_{r,-d,b}(a) \sum_{\substack{r' \bmod 2m \\ b' \bmod 2m|\Delta_0|}} e^{2\pi i\left(\frac{rr'\Delta_0 - bb'}{2m|\Delta_0|}\right)} \vartheta_{m,r'}(\tau, z)\, \vartheta_{m|\Delta_0|,b'}(\tilde{\tau}, z')$$

with $\varepsilon = i$ for $\Delta_0 < 0$ and $= 1$ otherwise. We thus have to show that

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{1}{2}}} \sum_{\substack{r \bmod 2m,\, b \bmod 2m|\Delta_0| \\ b \equiv rr_0 \bmod 2m}} \psi_{r,-d,b}(a) e^{2\pi i\left(\frac{rr'\Delta_0 - bb'}{2m|\Delta_0|}\right)} = \psi_{r',a,b'}(d)$$

if $b' \equiv r'r_0 \bmod 2m$ and $= 0$ otherwise. Inserting the defining formula for $\psi_{\cdot,\cdot,\cdot}(\cdot)$ and taking on both sides of the last equation the finite Fourier transform with respect to a modulo $|\Delta_0|$ all culminates in the identity

$$\frac{1}{2\varepsilon m|\Delta_0|^{\frac{3}{2}}} \sum_{\substack{r \bmod 2m,\, b \bmod 2m|\Delta_0| \\ a,\, c \bmod |\Delta_0| \\ b \equiv rr_0 \bmod 2m \\ r^2\Delta_0 \equiv b^2 - 4mac \bmod 4m|\Delta_0|}} \chi_{\Delta_0}([ma, b, c]) e^{2\pi i\left(\frac{rr'\Delta_0 - bb' + 2m(ac' + a'c)}{2m|\Delta_0|}\right)}$$

$$= \begin{cases} \chi_{\Delta_0}([ma', b', c']) & \text{if } \begin{bmatrix} b' \equiv r'r_0 \bmod 2m \text{ and} \\ r'^2\Delta_0 \equiv b'^2 - 4ma'c' \bmod 4m|\Delta_0| \end{bmatrix} \\ 0 & \text{otherwise .} \end{cases} \tag{3}$$

This can now be proved using standard Gauss sum identities and we leave this to the reader.

The proof of the corollary is the usual exercise in unfolding an integral. First of all we can assume that ϕ is holomorphic or antiholomorphic accordingly as Δ_0 is negative or positive (otherwise both sides of the identity to be proved are zero; with respect to the left hand side cf. the remark following the discussion of the Petersson scalar products in Sect. 2). But then we can write – using the formula immediately before the Corollary –

$$\left[\left(\frac{\sqrt{m}|\Delta_0|}{\pi\bar{\varepsilon}}\right)^{-1}\phi(\tau,z)\overline{\Theta_{\Delta_0,r_0}(\tau,z;t)}+\pi i\left(\frac{\Delta_0}{0}\right)\phi(\tau,z)\overline{T_{r_0}(\tau,z)}\right]e^{-4\pi m y^2/v}v^2$$

$$=\sum_{\Upsilon}\sum_{l\geqq 1}\sum_{s\in\mathbb{Z}}\left(\frac{\Delta_0}{l}\right)\frac{1}{l}\frac{\partial}{\partial t}\phi(\tau',z')\overline{\kappa_{\Delta_0 s^2,r_0 s}(\tau',z';lst)}e^{-4\pi m y'^2/v'}v'^2 .$$

Here we still use y and v for the imaginary parts of z and τ respectively. Furthermore Υ runs through a complete set of representatives for $\mathscr{J}(\mathbb{Z})_\infty\backslash\mathscr{J}(\mathbb{Z})$ and for each such Υ we use $(\tau',z')=\Upsilon\cdot(\tau,z)$, y' and v' denoting the imaginary parts of z' and τ' respectively. Now unfolding the integral of the right hand side, taken over a fundamental domain of $\mathfrak{H}\times\mathbb{C}$ modulo $\mathscr{J}(\mathbb{Z})$ with respect to the $\mathscr{J}(\mathbb{Z})$-invariant measure $dV=\frac{du\,dv\,dx\,dy}{v^3}$, we obtain the expression

$$\int\limits_{\mathscr{J}(\mathbb{Z})_\infty\backslash\mathfrak{H}\times\mathbb{C}}\sum_{l\geqq 1}\sum_{s\in\mathbb{Z}}\left(\frac{\Delta_0}{l}\right)\frac{1}{l}\frac{\partial}{\partial t}\phi(\tau,z)\overline{\kappa_{\Delta_0 s^2,r_0 s}(\tau,z;lst)}e^{-4\pi m y^2/v}v^2\,dV .$$

But

$$\kappa_{\Delta,r}(\tau,z;t)=e^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+|\Delta|}{4m}iv+rz\right)}e^{2\pi i\sigma\xi}e^{-\frac{\pi m\eta^2}{|\Delta|v}}$$

(with $\sigma=\operatorname{sign}(\Delta_0)$), a fundamental domain for $\mathfrak{H}\times\mathbb{C}$ modulo $\mathscr{J}(\mathbb{Z})_\infty$ is given by the set $\{(\tau,\lambda\tau+\mu)\,|\,0\leqq u,\mu\leqq 1,\ 0<v,\ \lambda\in\mathbb{R}\}$, and for $z=\lambda\tau+\mu$ one has $dV=\frac{du\,dv\,d\lambda\,d\mu}{v^2}$. Thus, carrying out the integration with respect to u and μ we obtain

$$\sum_{l\geqq 1}\sum_{s\in\mathbb{Z}}\left(\frac{\Delta_0}{l}\right)\frac{1}{l}\frac{\partial}{\partial t}C_\phi(\Delta_0 s^2,r_0 s)e^{-2\pi i\sigma sl\xi}I_{s,l}(\eta)$$

where

$$I_{s,l}(\eta)=\int\limits_0^\infty\int\limits_{-\infty}^{+\infty}e^{-4\pi\left(\frac{r_0^2+|\Delta_0|}{4m}s^2+r_0 s\lambda+m\lambda^2\right)v}e^{\left(\frac{-\pi m l^2\eta^2}{|\Delta_0|v}\right)}d\lambda\,dv$$

$$=\frac{1}{2\sqrt{m}}\int\limits_0^\infty e^{-\pi\left(\frac{|\Delta_0|s^2 v}{m}+\frac{ml^2\eta^2}{|\Delta_0|v}\right)}v^{-\frac{1}{2}}dv$$

$$=\frac{1}{2}\left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}}\int\limits_0^\infty e^{-\pi l\eta|s|\left(w+\frac{1}{w}\right)}w^{-\frac{1}{2}}dw$$

$$= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} e^{-2\pi l|s|\eta} \int_0^{\infty} e^{-\pi l|s|\eta\left(\sqrt{w} - \frac{1}{\sqrt{w}}\right)^2} d\sqrt{w}$$

$$= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} e^{-2\pi l|s|\eta} \int_{-\infty}^{+\infty} e^{-4\pi l|s|\eta \sinh^2\theta} e^{\theta} d\theta$$

$$= \left(\frac{l\eta}{|\Delta_0||s|}\right)^{\frac{1}{2}} e^{-2\pi l|s|\eta} \int_{-\infty}^{+\infty} e^{-4\pi l|s|\eta \sinh^2\theta} \cosh\theta \, d\theta = \frac{1}{2|\Delta_0|^{\frac{1}{2}}|s|} e^{-2\pi l|s|\eta}.$$

Observing that $\frac{\partial}{\partial t} e^{2\pi i(-\sigma s l\xi + l|s|i\eta)} = 0$ for $-\sigma s \le 0$, and that $C_\phi(\Delta, -r) = -C_\phi(\Delta, r)$ for skew-holomorphic ϕ (since $\phi(\tau, z) = -\phi(\tau, -z)$) we obtain the formula for the Fourier development of $\langle \phi | \Theta_{\Delta_0, r_0}(\cdot, \cdot; t) \rangle$ as given in the Corollary. Note that this formula shows in particular that $\langle \phi | \Theta_{\Delta_0, r_0}(\cdot, \cdot; t) \rangle$ is holomorphic. Since we saw in Sect. 2 that it is bounded by a polynomial in η independently of ξ we deduce that it must even be regular at the cusps. Thus, it is a modular form.

4. Proof of the lemma

For $t = i\eta$ one has $A^*t = -A\bar{t}$ and thus $f(A^*t)d(A^*t) = -f(-A\bar{t})d(A\bar{t})$. Decompose $f(t)$ as $f_+(t) + if_-(t)$ with $f_\pm(t) = \frac{1}{2\sqrt{\pm 1}}(f(t) \pm \overline{f(-\bar{t})})$. The modular forms $f_+(t)$ and $f_-(t)$ have real Fourier coefficients, i.e. satisfy $f_\pm(-\bar{t}) = \overline{f_\pm(t)}$, and hence $f_\pm(-A\bar{t})d(A\bar{t}) = \overline{f_\pm(At)d(At)}$. Thus, for $t = i\eta$, the differential $f(At)d(At) + \varepsilon f(A^*t)d(A^*t)$ equals $2\mathrm{Re}[f_+(At)d(At)] + 2i\mathrm{Re}[f_-(At)d(At)]$ if $\varepsilon = -1$ and equals the same expression but with 'Re' replaced by 'Im' and multiplied by i if $\varepsilon = +1$. We assume the first case, the other one can be treated similarly. The assumption about f then implies that for all $A \in SL_2(\mathbb{Z})$ both integrals $\int_0^{i\infty} \mathrm{Re}[f_\pm(At)d(At)]$ are absolutely convergent and equal to zero. Thus we may assume that f equals f_+ or f_-, or, more generally, that f itself has the property that all the integrals $\int_0^{i\infty} \mathrm{Re}[f(At)d(At)]$ converge absolutely and equal 0, and we have to show that $f(t)$ vanishes identically.

To prove this, consider $\varphi(B) := \int_{t_0}^{Bt_0} \mathrm{Re}[f(t)d(t)]$ for $B \in \Gamma_0(m)$ and $t_0 \in \mathfrak{H}$. Note that $\varphi(B)$ does not depend on the choice of t_0. Now let $A, B \in SL_2(\mathbb{Z})$ such that $ABA^{-1} \in \Gamma_0(m)$, let $B = \pm T^{n_1}ST^{n_2}S \dots T^{n_r}S$ with $n_j \in \mathbb{Z}$ and T, S denoting the generators $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ of $SL_2(\mathbb{Z})$ respectively, and set $B_j := \pm T^{n_1}ST^{n_2}S \dots T^{n_j}S$, $B_0 := 1$. Write $\varphi(ABA^{-1}) = \int_{t_1}^{Bt_1} \mathrm{Re}[f(At)d(At)]$ $(t_1 = A^{-1}t_0)$, $\int_{t_1}^{Bt_1} = \int_{t_1}^{B_1 t_1} + \int_{B_1 t_1}^{B_2 t_1} + \dots + \int_{B_{r-1} t_1}^{B_r t_r}$, and $\int_{B_j t_1}^{B_{j+1} t_1} \mathrm{Re}[f(At)d(At)] = \int_{t_1}^{T^{n_{j+1}}St_1} \mathrm{Re}[f(AB_jt)d(AB_jt)]$. Note that one has $T^{n_{j+1}}St_1 = -\frac{1}{t_1} + n_{j+1}$. Thus, setting $t_1 = i\eta$ and letting η tend to 0, it is easily deduced from the assumptions about

f, that $\int_{t_1}^{T^{n_{j+1}}St_1} \mathrm{Re}[f(AB_jt)d(AB_jt)] \to n_{j+1}f(AB_ji\infty)$. Here $f(s)$, for any rational number s or $s = i\infty$, denotes the constant term in the Fourier expansions of $f(t)$ at the cusp s. Summarizing, we have $\varphi(ABA^{-1}) = \sum_{j=0}^{r-1} n_{j+1}f(AB_ji\infty)$.

In particular, choosing $B = 1 = -TSTSTS$ in this identity and observing $\varphi(1) = 0$, we obtain $f(Ai\infty) + f(A1) + f(A0) = 0$ for all $A \in SL_2(\mathbb{Z})$. But this implies $f(s) = 0$ for all cusps s. Namely, write $s = \frac{\alpha}{\gamma}$ with relative prime integers α, γ, and choose integers β, δ such that $\alpha\delta - \beta\gamma = 1$. We can even choose β, δ such that δ and $\gamma + \delta$ are prime to m, except in the case m even and γ odd, where we choose β, δ such that $\gamma + \delta$ is prime to m and $\gcd(m, \delta) = 2$ (If a given solution δ of $\alpha\delta - \beta\gamma = 1$ has not these properties then choose an integer v such that $\delta + \gamma v \equiv -2\gamma \bmod m'$ (resp. $\bmod 2m'$ if m is even and γ is odd,) where m' is the product of all primes of m which do not divide γ, and replace δ, β by $\delta + \gamma v$, $\beta + \alpha v$). Setting $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we find $Ai\infty = s$, $A1 = \frac{\alpha + \beta}{\gamma + \delta}$, $A0 = \frac{\beta}{\delta}$, and, since $\gamma + \delta$, δ are prime to m (except for the case ...) the cusps $A1$, $A0$ are equivalent modulo $\Gamma_0(m)$ to the cusp 0 (except for the case ... where $A0$ is equivalent to $\frac{1}{2}$). Thus, we have $f(s) + 2f(0) = 0$ (or $f(s) + f(0) + f(\frac{1}{2}) = 0$ if m is even and γ is odd.) Since this equation holds for any s, we now deduce that $f(t)$ vanishes at the cusps.

But then we conclude that $\mathrm{Re}[f(t)dt]$ induces a harmonic differential on the compactification of $\Gamma_0(m)\backslash\mathfrak{H}$, which, by the above, satisfies $\int_{t_0}^{Bt_0} \mathrm{Re}[f(t)\,dt] = 0$ for all $B \in \Gamma_0(m)$ and all t_0. Hence the function $F(t) := \int_{t_0}^{t} \mathrm{Re}[f(t')dt']$ induces a harmonic function on this compact Riemann surface (to prove the invariance under $\Gamma_0(m)$, use $F(Bt) = F(t) + \int_{t_0}^{Bt_0} \mathrm{Re}[f(t')dt']$ for $B \in \Gamma_0(m)$), hence $F(t)$ is constant, hence $\mathrm{Re}[f(t)\mathrm{d}t] \equiv 0$, and since $f(t)$ is holomorphic, this finally implies that $f(t)$ vanishes identically.

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