Selmer groups, associated to vanishing of $L(f,s)$ at the central point $s = k - 1$, constructed by different methods by Skinner-Urban [SU] and Nékovař [N].

References

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.

[Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compositio Math.,* **143** (2007), 290–322.

[FvdG] C. Faber, G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II, *C. R. Math. Acad. Sci. Paris* **338** (2004), 381–384 and 467–470.

[Ha] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003. http://www.math.uni-bonn.de/people/harder/Manuscripts/Eisenstein/kolloquium.pdf

[Ka] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, preprint, 2005. http://eprints.math.sci.hokudai.ac.jp/archive/00000951/

[N] J. Nekovář, Selmer Complexes, *Astérisque* **310** (2006), 559 pp.

[SU] C. Skinner, E. Urban, Sur les déformations p-adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), 629–698.

[vdG] G. van der Geer, Siegel Modular Forms, preprint, 2006. arXiv:math.AG/0605346 v1

[We] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.

On the computation of modular forms of half-integral weight

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We indicate a method for a systematic and explicit generation of a basis for a given space of half integral modular forms on $\Gamma_0(4N)$ with arbitrary nebentype.

Preliminary remarks

The computational theory of elliptic modular forms of integral weight is meanwhile quite well understood. The most efficient method for calculating modular forms is based on the theory of modular symbols and goes back directly to [Ma]. It was used (with several improvements by various authors) by Henri Cohen, the author and Don Zagier to produce in the late 80's tables of modular forms [modi] and later by William Stein to produce another extensive database of modular forms [St]. This method is now implemented (mainly by William Stein) into the computer algebra systems MAGMA and SAGE.

Shimura showed [Sh] that modular forms of half-integral weight are intimately connected to forms of integral weight. This connection became even more important by Waldspurger's theorem relating values of the twisted L-series of newforms of integral weight at the critical point to the Fourier coefficients of associated half-integral weight forms. A famous application of these ideas to classical number theory is Tunnell's theorem on congruent numbers.

Despite their importance for the arithmetic theory of modular forms of integral weight there is no good algorithm to compute systematically half-integral weight forms. The main method used so far[1] was described in Basmaji's thesis [Ba,

[1]William Stein implemented this recently in SAGE.

pp. 55]: If N is divisible by 4, if $k = 2l - 1$ denotes an odd integer, and if we set $\theta_2 = \sum_{n\in 2\mathbb{Z}+1} q^{n^2}$ and $\theta_3 = \sum_{n\in\mathbb{Z}} q^{n^2}$ (where q, for z in the upper half plane, is the function $\exp(2\pi i z)$), then the image of the map[2]

$$S_{\frac{k}{2}}(4N,\chi) \to S_l\left(4N,\chi\chi_{-4}^l\right) \times S_l\left(4N,\chi\chi_{-4}^l\right), \quad f \mapsto (f\theta_2, f\theta_3)$$

equals the set of all pairs of modular forms (f_2, f_3) in $S_2(4N,\chi)$ which satisfy $f_2\theta_3 = f_3\theta_2$. This method is for example implemented in recent versions of SAGE. The disadvantages of this method lie at hand: It is based on a prior computation of modular forms of integral weight. It is not compatible in any sense with Hecke theory. Even if one is interested in only a single Hecke eigenform of half-integral weight one needs to compute first of all basis for the spaces $S_{\frac{3}{2}}(4N,\chi)$ and $S_2(4N,\chi)$, where the the assumption that N is divisible by 4 becomes then especially annoying.

We suggest a different approach, which overcomes the listed disadvantages. This approach is based on modular symbols. It allows to produce closed formulas for the Fourier coefficients of modular forms of half-integer weight in a very direct way. In fact, this idea behind this method is not really new. It was developed and used in [Sk1], [Sk2], [Sk3] to produce closed formulas for Jacobi forms (on the full modular group) of arbitrary weight and index. A more detailed account of this method will be published elsewhere [Sk4]. For simplicity we discuss in the following only the case of half-integral modular forms of weight $\frac{3}{2}$.

Statement of Results

The starting point to derive formulas for generators of a space $S_{\frac{3}{2}}(4N,\chi)$ are the following two theorems. Note that we use $S_{\frac{3}{2}}(4N,\chi)$ for the space of *non trivial cusp forms*, i.e. the orthogonal complement with respect to the Petersson scalar product of the space of all cusp forms f of weight $\frac{3}{2}$ on $\Gamma_0(4N)$ and with character χ which are linear combinations of theta series of the form $\sum_n n\psi(n)q^{tn}$ (t an integer and ψ a Dirichlet character).

Theorem 1 ([Sh])**.** *Let t be a positive squarefree integer. Then the application*

$$f = \sum_{n>0} c_f(n)\, q^n \mapsto \sum_{n>0}\sum_{d|n} (\chi\chi_{-4t})\,(n/d)\, a_f\left(td^2\right)\, q^n,$$

defines a map

$$S_{t,\chi} : S_{\frac{3}{2}}(4N,\chi) \to S_2(2N,\chi^2).$$

This map commutes with all Hecke operators $T(p)$ with $\gcd(p, 2N) = 1$.

(Note that the precise level $2N$ of the image of the maps $S_{t,\chi}$ was only conjectured in [Sh] and later proved in [Ni].)

[2]For a half-integral or integral integer k and a Dirichlet character χ we use $S_k(N,\chi)$ for the space of (non trivial, if $k = \frac{3}{2}$, see below) cusp forms on $\Gamma_0(N)$ of weight k and nebentype χ. If k is half-integral then N is assumed to be divisible by 4. For a discriminant D, we use χ_D for the Dirichlet character modulo D which, for odd primes p, equals the usual Legendre symbol $\left(\frac{D}{p}\right)$.

Theorem 2 ([Sk2]). *For every positive integer m, the application*

$$f \mapsto \lambda_f, \quad \lambda_f(c) := \sum_{s \in \mathbb{P}_1(\mathbb{Q})} c_s \left(\int_s^\infty + \int_{-s}^\infty \right) f(z)\, dz =: \int_{c^+} f$$

($c = \sum_s c_s(s) \in \mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$) defines an isomorphism

$$\pi : S_2(m, \chi) \to \frac{\mathrm{Hom}_{\Gamma_0(\mathrm{m})} \left(\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0, \mathbb{C}(\chi)\right)^{ev.}}{\mathrm{res}\,\mathrm{Hom}_{\Gamma_0(\mathrm{m})} \left(\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})], \mathbb{C}(\chi)\right)^{ev.}}.$$

This isomorphism commutes with all Hecke operators $T(p)$.

Here the notations are as follows. By $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$ we denote the free abelian group generated by elements (s), where s runs through the points of the rational projective line $\mathbb{P}_1(\mathbb{Q})$, and by $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$ we denote its subgroups of elements $c = \sum_s c_s(s)$ with $\sum_s c_s = 0$. The semigroup of regular integral 2×2 matrices acts on this group by linear extension of its natural action on $\mathbb{P}_1(\mathbb{Q})$. We use $\mathbb{C}(\chi)$ for the $\Gamma_0(m)$-module with underlying vector space $\mathbb{C}$ and the action[3] $([a, b, c, d], z) \mapsto \chi(d)z$. The vectors spaces whose quotient appears on the right of the claimed isomorphism are the spaces of all *even* $\Gamma_0(m)$-equivariant maps from $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$ (resp. $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$) into $\mathbb{C}(\chi)$. Here a map λ is called even if $\lambda\left(\sum c_s(-s)\right) = \lambda\left(\sum c_s(s)\right)$. The map res restricts a λ on $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$ to $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$. Finally, for a natural number l, the Hecke operator $T(l)$ is defined on each of the two spaces on the right by

$$\left(T(l)\lambda\right)(c) = \sum_{R=[a,b,c,d]} \chi(a)\lambda(Rc),$$

where R runs through a system of representatives for the set of integral 2×2 matrices $R = [a, b, c, d]$ of determinant l with c divisible by m modulo left multiplication by $\Gamma_0(m)$. (Note that $\chi(a) = 0$ unless a is relatively prime to m.)

The quotient on the right of the isomorphism of the last theorem, but with the restriction to even maps dropped, can be naturally identified with the dual of the space

$$C(m, \chi) := \ker\left(\left[\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0 \otimes_Z \mathbb{C}(\chi)\right]_{\Gamma_0(m)} \to \left[\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})] \otimes_Z \mathbb{C}(\chi)\right]_{\Gamma_0(m)} \right),$$

where the subscript $\Gamma_0(m)$ indicates that we consider the respective spaces of $\Gamma_0(m)$-coinvariants.

We now fix a natural number N and a squarefree natural number t and consider the composed map

$$L^*_{t,\chi} = \pi \circ S_{t,\chi} : S_{\frac{3}{2}}(4N, \chi) \to S_2(2N, \chi^2) \to C_{2N}(\chi^2)^*.$$

By dualising this map and identifying the dual space $S_{\frac{3}{2}}(4N, \chi)^*$ with the space $S_{\frac{3}{2}}(4N, \chi\chi_{4N})$ we obtain a map

$$L_{t,\chi} : C(2N, \chi^2) \to S_{\frac{3}{2}}(4N, \chi\chi_{4N}).$$

[3] We use $[a, b, c, d]$ for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The natural map $S_{\frac{3}{2}}(4N, \chi\chi_{4N}) \to S_{\frac{3}{2}}(4N, \chi)^*$ to which we refer is given explicitly by $f \mapsto \langle \cdot, \iota W_{4N} f\rangle$. Here ι and W_{4N} denote the maps $(\iota f)(z) = \overline{f(-\overline{z})}$ and $(W_{4N} f)(z) = f(-1/4Nz)\left(-i\sqrt{4n}\tau\right)^{-3/2}$, respectively, and we use the Petersson scalar product. It can be checked that then $L_{t,\chi}$ is in fact Hecke-equivariant (with respect to the Hecke operators $T(p)$ with p relatively prime to $4N$). Moreover, from the preceding explanations it is not hard to verify that the images of the $L_{t,\chi}$ with t running through all squarefree integers exhaust the space $S_{\frac{3}{2}}(4N, \chi\chi_{4N})$. Hence, for deducing from these considerations an effective algorithm to compute modular forms of weight $\frac{3}{2}$ in closed form we have to answer the question whether we are able to compute the $L_{t,\chi}$ in an explicit way.

By its very definition the map $L_{t,\chi}$ is defined by the identity

$$\langle g, \iota W_{4N} L_{t,\chi}(c)\rangle = \int_{c^+} S_{t,\chi}(g) \qquad (g \in S_{\frac{3}{2}}(4N, \chi).$$

Now, the maps $S_{t,\chi}$ are *theta liftings*, i.e. there exists a so-called theta kernel $\theta_{t,\chi}(z, \tau)$, which transforms in the first variable under $\Gamma_0(4N)$ like an element in $S_{\frac{3}{2}}(4N, \chi)$ and which transforms in the second variable under $\Gamma_0(2N)$ like an element in the space which is obtained from $S_2\left(2N, \chi^2\right)$ by taking the complex conjugates of its forms, and such that

$$S_{t,\chi}(g)(\tau) = \langle g, \theta_{t,\chi}(\cdot, \tau)\rangle$$

Explicit formulas for the theta kernels in question have be obtained in [Ni] and [Ci].

Inserting $\theta_{t,\chi}$ in the defining identity for $L_{t,\chi}$ we obtain after some obvious manipulations the formula

$$L_{t,\chi}(c) = W_{4N}\iota \int_{c^+} \theta_{t,\chi}(\cdot, \tau)\, d\overline{\tau}.$$

It is a priori not clear whether this is a sensible formula since $\theta_{t,\chi}$ is only real analytic and since we need to interchange taking scalar products and integration along hyperbolic lines. However, it can be verified by analyzing explicit expressions for $\theta_{t,\chi}$ that this formula holds in fact true.

It turns out that the right hand side of the last formula can indeed be computed explicitly and is, moreover, given by a simple and appealing combinatorial formula. The method of computation is in essence the same as in [Sk1]. Details have been worked out (for special cases) by Reinhard Steffens [Sts] in his diploma thesis. We state the final result here in a slightly weaker form.

Theorem 3 ([Sts], [Sk4])**.** *For natural numbers D whose squarefree part does not divide $4Nt$, the D-th Fourier coefficient of $L_{t,\chi}\left(\sum_s c_s\,(s)\right)$ is given by*

$$\sum_{Q \in F_N(4NDt)} \chi_t(Q) \sum_s c_s \operatorname{sign}\left(Q(s)\right).$$

Here, for a natural number Δ, we use $F_N(\Delta)$ for the set of all binary integral quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ such that $b^2 - 4ac = \Delta$ and such that $N|a$ and $2N|b$. Moreover, for any such form, we use $\chi_t(Q) = \chi(a/N)\chi_{-4t}(a/N)$

(in particular $\chi_t(Q) = 0$ *if* $\gcd(a, 4N) \neq 1$*) and* $\operatorname{sign} Q(s)$ *for the sign of* $Q(x, y)$ *if* $s = [x : y]$.

We leave it to the reader to verify that the inner sum in the given formula is different from 0 for only finitely many Q. The assumption on D can be dropped for the cost of adding certain more complicated terms to the given formula. The given formula can be interpreted in terms of intersection numbers of certain cycles on the modular curve $X_0(2N)$. We finally mention that a similar theorem (with suitable modifications) holds true for arbitrary half integral weight. Details and proofs will be published elsewhere [Sk4].

References

[Ba] J. Basmaji, Ein Algoritmus zur Berechnung von Hecke-Operatoren und Anwendungen auf modulare Kurve, Dissertation, Universität Essen 1996

[Ci] B. Cipra, On the Niwa-Shintani theta kernel lifting of modular forms. Nagoya Math. J. 91 (1983), 49–117

[Ma] J.I. Manin, Parabolic points and zeta functions of modular curves, Izv. Akad. Nauk SSSR 36 (1972), 19–66

[modi] http://modi.countnumber.de

[Ni] S. Niwa, Modular forms of half integral weight and the integral of certain theta functions. Nagoya Math. J. 56 (1974), 147–161

[Sh] G. Shimura, On modular forms of half-integral weight. Ann. of Math. 97 (1973), 440–481

[Sk1] N-P. Skoruppa, Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms. Invent. Math. 102 (1990), 501–520

[Sk2] N-P. Skoruppa, Heegner Cycles, Modular Forms and Jacobi Forms. Sém. Th. des Nombres Bordeaux 3 (1991), 93–116

[Sk3] N-P. Skoruppa, Binary quadratic forms and the Fourier coefficients of elliptic and Jacobi modular forms, J. Reine Angew. Math. 411 (1990), 66–95

[Sk4] N-P. Skoruppa, Explicit Formulas for the Fourier Coefficients of Modular Forms of Half Integral Weight, in preparation

[St] W. Stein, http://modular.fas.harvard.edu/tables/

[Sts] R. Steffens, Explizite Formeln für Fourier-Koeffizienten von Modulformen vom Gewicht $\frac{3}{2}$. Diplomarbeit, Universität Siegen 2006

Even unimodular lattices with a complex structure and their theta series

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Let K be an imaginary quadratic number field with class number 1, ring of integers $\mathcal{O}_K$ and discriminant D_K. We consider $\mathbb{C}^k$, $k \in \mathbb{N}$, with the standard Hermitian scalar product. The lattices Λ under consideration are given by $\mathbb{C}$-linearly independent vectors

$$b_1, \ldots, b_k \in \mathbb{C}^k \quad \text{with Gram matrix} \quad S = (\langle b_\nu, b_\mu \rangle)$$

satisfying

$$\Lambda = \mathcal{O}_K b_1 + \ldots + \mathcal{O}_K b_k, \quad \langle \lambda, \lambda \rangle \in 2\mathbb{Z} \text{ for all } \lambda \in \Lambda, \quad \det S = \left(\frac{2}{\sqrt{|D_K|}} \right)^k .$$