Dear Andreea,

Appended you find a scan of those pages of your thesis which I annotated. However, here is a list of those things which I definitly would like you to change to conform to correct scientific publishing:

1. All theorems which are not due to you should be marked clearly as cited from the literature.

This is usually done as follows in LaTeX:

```
\begin{Theorem}[\protect{\cite[Thm.5, p.24]{Serre63}}] ....\end{Theorem}
```

It does not suffice to give the reference say somewhere 5 lines before or after the theorem as you do. (Some people like Zagier even insist that theorems which are not due to the author are not marked as theorem, butI find a this bit extreme.)

So please go through your thesis with a "search and find" and correct all theorem statements accordingly.

2. The "Appendix" should be placed before the Bibliography. (This is usually achieved by using macros like \frontmatter, \backmatter etc. in the LaTeX book styles.)

3. It is not really necessary but I propose emphatically to have the tags in the bibliography not as numbers [1], [2] etc. but as [Aj], [Brm], [Boy] etc. For a reader of your thesis it is completely meaningless when you say "it was proved in [5] ...", but much more informative if he reads "it was proved in [Boy] ..." (since then often he has not to look up in the bibliography but can make a guess, thereby saving time).

4. If you did not use Bibtex then please use it in your revised version. This

a) makes sure that only those items show up in the bibliography which are really cited (which is absolutely mandatory for a scientific publication with very few exceptions like a compilation of references, of course).

b) simplifies it to apply a style which uses name abbreviations as tags instead of numbers (see remark before).

5. The table of Hecke eigenvalues should be replaced by the Fourier coefficients of the Jacobi forms. (I think Ali also did not do this and I regret that I missed to point this out to him.)

6. The two subsub...sections "1.2.4.2 Jacobi forms and Siegel modular forms" and "1.2.4.4 Jacobi forms and orthogonal modular forms" should be suppressed since

a) they do not contain any discovery or notion due to you,

b) they do not contain any discovery or notion which needs to be cited to understand your main results and their proofs,

c) they contain very wrong statements which in view of a) and b) are not worth to be discussed and corrected now (see the scans which follow if you are interested or ask me at a later point).

So, please throw them out.

7. Suppress all text from page 91, line 6 (i.e. "We introduce some additional notions from [41]" until page 95, line -7 (i.e. until the end ofSections 3.2). The reason is

a) these pages do not contain any discovery or notion due to you,

b) they do not contain any discovery or notion which needs to be cited to understand your main results and their proofs,

c) these pages reflect genuine research work of a book fragment of myself which will be eventually published in some book series; I gave this fragment a while ago to Fredrik with the explicit request "not to distribute it". It is more or less OK that he shared those insights from the fragment with you while advising your thesis, but it is definitely not OK to include them into your thesis without my allowance (which anyway I would not have given because of b)).

I am convinced that these revisions will make your thesis much nicer and I hope that the accompanying comments are helpful for your future publications.

One last point which, as I hope, will also be helpful for you in the future: At various places there are quite general and somewhat bold statements in the thesis which make readers unhappy: "Orthogonal modular forms have many applications in algebraic geometry" (p. 30) or "Jacobi Eisenstein series ... were used to develop a theory of newforms" (p. 1). These make unhappy because they give no real information (since they are too vague) and because they do not necessarily agree with the understanding of the reader or since they are even objectively wrong (like the second example). Such remarks do not belong into a scientific publication (which tries to discover facts and truth, but should not propose subjective interpretations or assessments)! They are acceptable in a talk (since the audience has the chance to correct a wrong point of view) and maybe not avoidable in an article for a general audience. Asides, such statements often have a bad impact: A young thesis student 10 years from now might read some of your statements, and since they are 10 year old and in an officially accepted thesis they grew in weight over time: so he takes them as true and propagates them by himself. This kind of game led often to complete wrong and partly ridiculous assessments of historical facts in Mathematics as every senior mathematician knows from many examples.

In the following pages there are a few more minor corrections (and maybe some of the above repeated).

Best,
---Nils

CHAPTER 1

Introduction

Interest in Jacobi forms has increased in recent years due to their numerous applications to number theory, algebraic geometry and string theory. Computing Jacobi forms gives direct information on the Fourier coefficients of half-integral weight modular forms [**29**], they play a part in the Mirror Symmetry conjecture for K3 surfaces [**16**] and a certain type of Jacobi forms can be the elliptic genus of Calabi–Yau manifolds [**19**], to name some of these applications.

Our long-term goal is to determine the precise relation between Jacobi forms of lattice index and elliptic modular forms. This would enable the transfer of information and mathematical tools from one side to the other. Lifts of Jacobi forms to other type of automorphic forms often have special properties, for example their Fourier coefficients satisfy simple linear relations [**26**], or their L-functions satisfy certain vanishing properties [**14**]. An intermediate step towards our goal is to develop a theory of newforms for Jacobi forms of lattice index. While the term "newform" is usually applied to cusp forms, it is important to define this for Eisenstein series as well, in order to obtain a complete description.

1.1. Statement of results

1.1.1. Poincaré and Eisenstein series. Eisenstein and Poincaré series are the most simple examples of modular forms. They are obtained by taking the average of a function over a group (modulo a parabolic subgroup) and hence are invariant under the group action by construction. In the context of elliptic modular forms, it is well-known that they satisfy the important property of reproducing Fourier coefficients of cusp forms under a suitably defined scalar product. Furthermore, Poincaré series generate the space of elliptic cusp forms [**10**, §8.2].

Jacobi–Poincaré series of matrix index were used to construct lifting maps between spaces of Jacobi cusp forms and subspaces of elliptic modular forms [**6**]. Jacobi–Eisenstein series of scalar index were used to develop a theory of newforms, which does not exist yet for arbitrary lattice index [**14**].

To the best of the author's knowledge, Poincaré series have not been defined in the literature for Jacobi forms of lattice index. Let k be a positive integer and let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ (see Subsection 1.2.2). For every pair (D, r) in the support of $\underline{L}$ (1.12) such that $D < 0$, define the Poincaré series of weight k and index $\underline{L}$ associated with the pair (D, r) as the series (2.1).

THEOREM A. *The Poincaré series $P_{k,\underline{L},D,r}$ satisfies the following:*

(i) If $k > \mathrm{rk}(\underline{L}) + 2$, then $P_{k,\underline{L},D,r}$ is absolutely and uniformly convergent on compact subsets of $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ and it is a Jacobi cusp form of weight k and index $\underline{L}$. Furthermore, it reproduces the Fourier coefficients of Jacobi cusp forms of the same weight and index under the Petersson scalar product (1.23).

(ii) *The Poincaré series $P_{k,\underline{L},D,r}$ has the following Fourier expansion:*

$$P_{k,\underline{L},D,r}(\tau,z) = \sum_{\substack{(D',r')\in supp(\underline{L})\\ D'<0}} G_{k,\underline{L},D,r}(D',r')e\left((\beta(r')-D')\tau+\beta(r',z)\right),$$

where

$$\begin{aligned} G_{k,\underline{L},D,r}(D',r') :=& \delta_{\underline{L}}(D,r,D',r') + (-1)^k\delta_{\underline{L}}(D,-r,D',r') + \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}} \\ &\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} \sum_{c\geq 1} c^{-\frac{\mathrm{rk}(\underline{L})}{2}-1} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right) \\ &\times \left(H_{\underline{L},c}(D,r,D',r') + (-1)^k H_{\underline{L},c}(D,-r,D',r')\right), \end{aligned}$$

the function $\delta_{\underline{L}}(D,r,D',r')$ is defined in (2.5), *the function J_α is the J-Bessel function of index α and $H_{\underline{L},c}(D,r,D',r')$ is defined in* (2.6).

As a consequence, the set

$$\{P_{k,\underline{L},D,r} : r\in L^\#/L, D\in\mathbb{Q}_{<0} \text{ and } \beta(r)\equiv D \bmod \mathbb{Z}\}$$

generates the $\mathbb{C}$-vector space of Jacobi cusp forms of weight k and index $\underline{L}$.

The definition of Jacobi–Eisenstein series of lattice index was given for instance in [**1**], where some of their properties were studied (such as dimension formulas for their spanning set and the fact that they are Hecke eigenforms). For every r in $L^\#$ such that $\beta(r)\in\mathbb{Z}$, the Eisenstein series of weight k and index $\underline{L}$ associated with r is defined as the series (1.14).

Theorem B. *The Eisenstein series $E_{k,\underline{L},r}$ satisfies the following:*

(i) *If $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$, then $E_{k,\underline{L},r}$ is absolutely and uniformly convergent on compact subsets of $\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C})$ and it is a Jacobi form of weight k and index $\underline{L}$. Furthermore, it is orthogonal to cusp forms of the same weight and index under the Petersson scalar product.*

(ii) *The Eisenstein series $E_{k,\underline{L},r}$ has the following Fourier expansion:*

$$\begin{aligned} E_{k,\underline{L},r}(\tau,z) =& \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k\vartheta_{\underline{L},-r}(\tau,z)\right) \\ &+ \sum_{\substack{(D',r')\in supp(\underline{L})\\ D'<0}} G_{k,\underline{L},r}(D',r')e\left((\beta(r')-D')\tau+\beta(r',z)\right), \end{aligned}$$

where $\vartheta_{\underline{L},r}$ is a theta series as in (1.17),

$$\begin{aligned} G_{k,\underline{L},r}(D',r') :=& \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k}{2\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)} \times (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \\ &\times \sum_{c\geq 1} c^{-k}\left(H_{\underline{L},c}(r,D',r') + (-1)^k H_{\underline{L},c}(-r,D',r')\right) \end{aligned}$$

and $H_{\underline{L},c}(r,D',r')$ is defined in (2.14).

As a consequence, the series $E_{k,\underline{L},r}$ only depends on r modulo L. We would like to obtain a closed formula for the Fourier coefficients of Eisenstein series. Lattice sums similar to $H_{\underline{L},c}(r,D',r')$ also arise in the Fourier expansions of Poincaré and Eisenstein series for vector-valued modular forms and those of orthogonal modular forms [**7, 43**], as well as in trace formulas for these types of automorphic forms [**39, 25**]. Most of the literature deals with the simplest case, which is equivalent to taking $r=0$ in $L^\#/L$. Even for Jacobi forms of scalar index, the authors of [**14**] compute the Fourier expansion of $E_{k,m,0}$ and state that "(the calculation) is tedious (for arbitrary r)". The mathematical objects that arise in these calculations are Gauss sums for abelian groups and representation numbers for quadratic forms. For an introduction to these topics, the reader can consult [**12**] and [**32**, §5], respectively, for example. In Section 2.3, we show that $E_{k,\underline{L},0}$ vanishes identically when k is odd and that, when k is even and greater than $\frac{\mathrm{rk}(\underline{L})}{2} + 2$, the following holds:

THEOREM C. *Let the pair (D', r') in the support of $\underline{L}$ be such that $D' < 0$. If the rank of $\underline{L}$ is even, then*

$$G_{k,\underline{L},0}(D',r') = \frac{2(-1)^{\lceil \frac{\mathrm{rk}(\underline{L})}{4} \rceil}(-D'|\mathfrak{d}|)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\mathfrak{f}L\left(1-k+\frac{\mathrm{rk}(\underline{L})}{2},\chi_{\mathfrak{d}}\right)\sum_{d|\mathfrak{f}}\mu(d)\chi_{\mathfrak{d}}(d)d^{\frac{\mathrm{rk}(\underline{L})}{2}-k}\sigma_{1-2k+\mathrm{rk}(\underline{L})}\left(\frac{\mathfrak{f}}{d}\right)} \times \prod_{p|2\tilde{D}'\det(\underline{L})} \frac{\tilde{L}_p(k-1)}{1-\chi_{\underline{L}}(p)p^{\frac{\mathrm{rk}(\underline{L})}{2}-k}}$$

and, if the rank of $\underline{L}$ is odd, then

$$\begin{aligned} G_{k,\underline{L},0}(D',r') = & \frac{\chi_8(\mathrm{rk}(\underline{L}))\, 2^{2k-\mathrm{rk}(\underline{L})}\left(\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - k\right)(D'\tilde{D}'_{0,r'})^{\frac{1}{2}}(-D')^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil-1}}{B_{2k-\mathrm{rk}(\underline{L})-1}\mathfrak{f}_{D',r'}|\mathfrak{d}_{D',r'}|^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil}} \\ & \times L\left(1-k+\lceil \tfrac{\mathrm{rk}(\underline{L})}{2} \rceil, \chi_{\mathfrak{d}_{D',r'}}\right) \sum_{d|\mathfrak{f}_{D',r'}} \mu(d)\chi_{\mathfrak{d}_{D',r'}}(d)d^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - k} \\ & \times \sigma_{2-2k+\mathrm{rk}(\underline{L})}\left(\frac{\mathfrak{f}_{D',r'}}{d}\right) \prod_{p|\tilde{D}'\det(\underline{L})} \frac{1-\chi_{\underline{L}}(\tilde{D}'_{0,r'},p)p^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil-k}}{1-p^{1-2k+\mathrm{rk}(\underline{L})}}\tilde{L}_p(k-1). \end{aligned}$$

The quantities appearing in the above theorem are defined in Sections 1.2 and 2.3. For every r in $L^{\#}$, let N_r denote its order in $L^{\#}/L$. The Fourier coefficients of arbitrary Eisenstein series satisfy the following:

PROPOSITION D. *Suppose that $r \in L^{\#}/L$ and $\beta(r) \in \mathbb{Z}$. Then*

$$\sum_{m \in \mathbb{Z}_{(N_r)}} G_{k,\underline{L},mr}(D',r') = \begin{cases} \sum_{m \in \mathbb{Z}_{(N_r)}} G_{k,\underline{L},0}(D',r'+rm), & \text{if } \beta(r,r') \in \mathbb{Z} \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We use this result to compute the Fourier coefficients of Eisenstein series associated with elements of small order in Examples 2.26–2.29.

1.1.2. Hecke operators and the action of the orthogonal group. Hecke operators give extra structure to spaces of automorphic forms and they have algebraic interpretations in terms of the underlying surfaces. They can be used to construct equivariant lifting maps between different types of automorphic forms. Hecke operators acting on Jacobi forms of lattice index were defined in [**1**, §2.5] as double coset operators (Definition 3.2). It was shown there that they preserve spaces of Jacobi forms of fixed weight and index and that they are Hermitian under the Petersson scalar product. Their action on the Fourier coefficients of Jacobi forms was computed and their multiplicative properties were studied. Furthermore, by studying the L-functions attached to Hecke eigenforms, a relation between Jacobi forms and elliptic modular forms was formulated (Remarks 3.15 and 3.16). Explicit lifting maps were also defined in some cases and we discuss them in Subsection 3.1.2.

The discriminant module of $\underline{L}$ is the pair $D_{\underline{L}} = (L^{\#}/L, \beta \bmod \mathbb{Z})$. It is a finite quadratic module (Definition 1.11). It was shown in [**1**, §3.1] that the orthogonal group of $D_{\underline{L}}$ acts on Jacobi forms of weight k and index $\underline{L}$ from the left (Proposition 3.26).

Proposition E. *The operators arising from the action of the orthogonal group of $D_{\underline{L}}$ are unitary with respect to the Petersson scalar product.*

In particular, since these operators commute with Hecke operators and the spaces of Jacobi cusp forms of weight k and index $\underline{L}$ are finite-dimensional, every such space has a basis of common eigenforms. Furthermore, the orthogonal group of $D_{\underline{L}}$ acts on Eisenstein series in the following way:

Proposition F. *For every s in the orthogonal group of $D_{\underline{L}}$,*

$$W(s)E_{k,\underline{L},r} = E_{k,\underline{L},s^{-1}(r)}.$$

In the case of lattices of rank one, reflection maps in the orthogonal group of $D_{\underline{L}}$ act on Jacobi forms in the same way that Atkin–Lehner involutions act on elliptic modular forms (Example 3.35):

Was L_m explained before?

Proposition G. *For every positive-definite, even, scalar lattice $\underline{L}_m$, the following equality holds:*

$$\left\{W(s_a) : s_a \text{ is a reflection map in } O(L_m^{\#}/L_m)\right\} = \{W_t : t \parallel m\}.$$

The root lattices D_n are defined in Example 1.6, (3). In Section 3.3, we compute the Hecke eigenvalues of Jacobi cusp forms of weight k and index D_n for small values of k and odd n. These eigenvalues are listed in Appendix A. We compare them with the eigenvalues of elliptic modular forms in Table 3.1, in order to verify the conjectured correspondence between Jacobi forms of odd rank lattice index and elliptic modular forms from [**1**].

Actually they are defined in the literature since 80 years

1.1.3. Level raising operators. Level raising operators are intimately connected to the theory of newforms. They can also be used to define additive lifting maps between Jacobi forms and other type of automorphic forms [**9, 26**]. Level raising operators of type $U(\cdot)$ arise from isometries of lattices (Definition 4.1).

Theorem H. *Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be two positive-definite, even lattices over $\mathbb{Z}$, such that $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ as modules over $\mathbb{Q}$ and there exists an isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Then $U(\sigma)$ maps Jacobi forms of weight k and index $\underline{L}_2$ to Jacobi forms of weight k and index $\underline{L}_1$. Furthermore, if ϕ has a Fourier expansion of the type*

$$\phi(\tau, z_2) = \sum_{(D,r_2) \in \operatorname{supp}(\underline{L}_2)} C_\phi(D, r_2) e\left((\beta_2(r_2) - D)\tau + \beta_2(r_2, z_2)\right),$$

then $U(\sigma)\phi$ has the following Fourier expansion:

$$U(\sigma)\phi(\tau, z_1) = \sum_{\substack{(D,r_1) \in \operatorname{supp}(\underline{L}_1) \\ \sigma(r_1) \in L_2^{\#}}} C_\phi(D, \sigma(r_1))\left((\beta_1(r_1) - D)\tau + \beta_1(r_1, z_1)\right).$$

As a corollary, the operators $U(\cdot)$ preserve cusp forms. If $\underline{L}_1$ and $\underline{L}_2$ are as above, then $(\sigma(L_1), \beta_2)$ is a sublattice of $\underline{L}_2$ and $\sigma : \underline{L}_1 \to (\sigma(L_1), \beta_2)$ is an isomorphism of lattices. Conversely, every sublattice (M, β_2) of $\underline{L}_2$ gives rise to an isometry of (M, β_2) into $\underline{L}_2$ given by inclusion. In other words, the above theorem asserts that, given a positive-definite, even lattice $\underline{L}$, for every overlattice $\underline{L}'$ of $\underline{L}$, Jacobi forms of weight k and index $\underline{L}'$ are Jacobi forms of weight k and index $\underline{L}$. Every Jacobi form of index $\underline{L}'$ is called an oldform of index $\underline{L}$. We obtain the following criterion:

LEMMA I. *Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$. If*

$$\phi(\tau, z) = \sum_{(D,r)\in \operatorname{supp}(\underline{L})} C_\phi(D, r) e((\beta(r) - D)\tau + \beta(r, z))$$

is a Jacobi form of weight k and index $\underline{L}$ such that $C_\phi(D, r) = 0$ for all r not in $L'^{\#}$ for some overlattice $\underline{L}'$ of $\underline{L}$, then ϕ is an oldform.

Level raising operators of type $V(\cdot)$ were constructed in [**18**] as the images of elliptic Hecke operators under a certain homomorphism of Hecke algebras, using the relation between Jacobi forms and orthogonal modular forms. The reader can also consult Definition 4.25 for a classical approach.

THEOREM J. *For every l in $\mathbb{N}$, the operator $V(l)$ maps Jacobi forms of weight k and index $\underline{L} = (L, \beta)$ to Jacobi forms of weight k and index $\underline{L}(l) := (L, l\beta)$. Furthermore, if ϕ has a Fourier expansion of the type*

$$\phi(\tau, z) = \sum_{(D,r)\in \operatorname{supp}(\underline{L})} C_\phi(D, r) e\left((\beta(r) - D)\tau + \beta(r, z)\right),$$

then $V(l)\phi$ has the following Fourier expansion:

$$V(l)\phi(\tau, z) = \sum_{(D,r')\in \operatorname{supp}(\underline{L}(l))} \sum_{\substack{a|(\beta(r)-D,l)\\ \frac{r'}{a}\in L(l)^{\#}}} a^{k-1} C_\phi\left(\frac{l}{a^2}D, \frac{l}{a}r'\right) e\left((\beta(r') - D)\tau + l\beta(r', z)\right).$$

As a corollary, the operators $V(\cdot)$ also preserve cusp forms. Level raising operators satisfy the following commutative properties:

PROPOSITION K (see Lemmas 4.32–4.34). *The operators $U(\cdot)$ and $V(\cdot)$ commute with each other.*

THEOREM L (see Theorems 4.36 and 4.37). *The operators $U(\cdot)$ and $V(\cdot)$ commute with Hecke operators and with the action of well-defined reflection maps.*

Level raising operators preserve Eisenstein series:

PROPOSITION M. *For every overlattice $\underline{L}'$ of $\underline{L}$, r in $L'^{\#}/L'$ such that $\beta(r) \in \mathbb{Z}$ and l in $\mathbb{N}$, the following holds:*

$$V(l) \circ U(L'/L) E_{k,\underline{L}',r} = \sum_{\substack{x\in L(l)^{\#}/L\\ l\beta(x)\in\mathbb{Z}}} \sum_{\substack{a|(l\beta(x),l)\\ r\equiv\frac{lx}{a} \bmod L'}} a^{k-1} E_{k,\underline{L}(l),x}.$$

It was shown in [**1**, §3.3] that "twisted" Eisenstein series (Definition 1.35) form a basis of Hecke eigenforms for Hecke operators (Theorem 3.11). We obtain a sufficient condition for twisted Eisenstein series to be oldforms in Theorem 4.40.

1.2. Preliminaries

This section contains the notation and elementary theory which are necessary in order to make the results in this thesis precise. We recall the definition of Jacobi forms of lattice index, following [**1**]. We discuss the connection between Jacobi forms and various other types of modular forms, with references from the literature. Finally, we list some examples.

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the set of positive natural numbers, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Set $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. The ring of integers modulo n is denoted by $\mathbb{Z}_{(n)}$.

Consider the branch of the complex square root with argument in $(-\pi/2, \pi/2]$. It follows that the function $z \mapsto \sqrt{z}$ takes positive reals to positive reals, complex numbers in the upper half-plane to the first quadrant and complex numbers in the lower half-plane to the fourth quadrant. Set $z^{k/2} := (\sqrt{z})^k$ when $k \in \mathbb{Z}$. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $\Re(z)$ and $\Im(z)$ denote its real and imaginary parts, respectively.

Let n in $\mathbb{Z}$ have prime factorization $up_1^{e_1} p_k^{e_k}$, with $u = \pm 1$, and let $a \in \mathbb{Z}$. The *Kronecker symbol* $\left(\frac{a}{n}\right)$ is defined as

$$\left(\tfrac{a}{n}\right) := \left(\tfrac{a}{u}\right) \prod_{i=1}^{k} \left(\tfrac{a}{p_i}\right)^{e_i}.$$

For an odd prime p, the number $\left(\frac{a}{p}\right)$ is the usual Legendre symbol and, when $p = 2$, it is equal to 0 when a is even, to 1 when $a \equiv \pm 1 \bmod 8$ and to -1 when $a \equiv \pm 3 \bmod 8$. Define $\left(\frac{a}{1}\right)$ to be equal to 1 and $\left(\frac{a}{-1}\right)$ to be equal to $\operatorname{sign}(a)$.

For every prime number p, the p-adic valuation for $\mathbb{Q}$ is defined as

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, v_p(n) := \begin{cases} \max\{v \in \mathbb{N}, p^v \mid n\}, & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ v_p(a) - v_p(b), & \text{if } n = \frac{a}{b} \in \mathbb{Q} \setminus \{0\} \text{ and} \\ \infty, & \text{if } n = 0. \end{cases}$$

The greatest common divisor of two integers a and b is denoted by (a, b). Write $b \parallel a$ if $b \mid a$ and $(b, \frac{a}{b}) = 1$. In sums of the form $\sum_{b|a}$ or $\sum_{ab=n}$, the summation is over positive divisors only. For an integer n, set $e_n(x) := e^{2\pi i x/n}$ and $e^n(x) := e^{2\pi i n x}$. Write $e(x) = e_1(x)$.

The J-Bessel function of index $\alpha > 0$ is defined by the following series expansion around $x = 0$:

$$J_\alpha(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n+\alpha}. \tag{1.1}$$

For every c in $\mathbb{N}$ and m, n in $\mathbb{Z} \setminus \{0\}$, define the Kloosterman sum

$$S(m, n; c) := \sum_{a \in \mathbb{Z}_{(c)}^{\times}} e_c(ma + na^{-1}), \tag{1.2}$$

where a^{-1} denotes the inverse of a modulo c.

Let $\mathbb{Z}_p$ denote the p-adic integers and let $\parallel \cdot \parallel_p$ be the p-adic norm, i.e. $\parallel a \parallel_p := p^{-v_p(a)}$.

Definition 1.1 (Igusa zeta function). Let $f \in \mathbb{Z}_p[X_1, \ldots, X_e]$. The Igusa zeta function of f at p is defined for every s in $\mathbb{C}$ with $\Re(s) > 0$ as the p-adic integral

$$\zeta(f; p; s) := \int_{\mathbb{Z}_p^e} \parallel f(x) \parallel_p^s \, dx.$$

It was proved in [23] that $\zeta(f; p; s)$ is a rational function in p^{-s} and hence it has a meromorphic continuation to all of $\mathbb{C}$.

Let $\omega(\cdot)$, $\mu(\cdot)$, $\sigma_t(\cdot)$ and $\zeta(\cdot)$ denote the function counting the number of prime divisors of an integer, the Möbius function, the t-th divisor sum and the Riemann zeta function, respectively. We define $\sigma_t(n) = 0$ for n in $\mathbb{R} \setminus \mathbb{N}$. Let B_n denote the n-th Bernoulli number and define the n-th Bernoulli polynomial

$$B_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j. \tag{1.3}$$

We remind the reader of the following well-known identity:

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & n = 1 \text{ and} \\ 0, & \text{otherwise.}\end{cases} \tag{1.4}$$

Let R be a ring. The set of $n \times n$ matrices with entries in R is denoted by $\mathrm{M}_n(R)$. A matrix A in $\mathrm{M}_n(\mathbb{Z})$ is called even if it has even diagonal entries. Denote the group of invertible matrices in $\mathrm{M}_n(R)$ by $\mathrm{GL}_n(R)$, the group of matrices with positive determinant by $\mathrm{GL}_n^+(R)$ and the group of matrices with determinant equal to one by $\mathrm{SL}_n(R)$. For every $n \times m$ matrix A, its transpose is denoted by A^t.

Let $N \geq 1$ be an integer. A *Dirichlet character modulo* N is a map $\chi : \mathbb{Z} \to \mathbb{C}$ which satisfies the following properties:

- $\chi(x + N) = \chi(x)$ for all x in $\mathbb{Z}$,
- $\chi(x) = 0$ if and only if $(x, N) > 1$,
- $\chi(xy) = \chi(x)\chi(y)$ for all x, y in $\mathbb{Z}$.

For every Dirichlet character χ, let σ_t^χ denote the twisted divisor sum

$$\sigma_t^\chi(n) := \sum_{d|n}\chi(d)d^t$$

and, for every two Dirichlet characters ξ and χ, set

$$\sigma_t^{\xi,\chi}(n) := \sum_{d|n}\xi\left(\frac{n}{d}\right)\chi(d)d^t.$$

The *Dirichlet L-function* of a Dirichlet character χ is

$$L(s,\chi) := \sum_{n=1}^{\infty}\chi(n)n^{-s} = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}.$$

For every positive integer N, set

$$L_N(s,\chi) := \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty}\chi(n)n^{-s} = L(s,\chi)\prod_{p|N}\left(1 - \chi(p)p^{-s}\right).$$

A *discriminant* is an integer which is congruent to 0 or 1 modulo 4. For every discriminant D, the function $\chi_D := \left(\frac{D}{\cdot}\right)$ is a well-defined quadratic Dirichlet character and we set $L_D(\cdot) := L(\cdot, \chi_D)$. A *fundamental discriminant* is an integer d such that either $d \equiv 1 \bmod 4$ and d is square-free or $d = 4n$ for some n in $\mathbb{Z}$ such that $n \equiv 2$ or $3 \bmod 4$ and n is square-free.

Definition 1.2 (Conductor). Let ξ and χ be two Dirichlet characters modulo F and N, respectively, with $F \mid N$. If $\chi(n) = \xi(n)$ for every n in $\mathbb{Z}_{(N)}^\times$, then χ is *induced* by ξ. If χ is not induced by any character other than itself, then it is called *primitive*. It is well-known that every Dirichlet character χ is induced by a primitive Dirichlet character which is uniquely determined by χ. The conductor of χ is the period of the primitive character which induces it.

Let χ be a primitive character modulo N and define the Gauss sum

$$G(\chi) := \sum_{n=1}^{N}\chi(n)e_N(n)$$

and the constant

$$a_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \text{ and} \\ 1, & \text{if } \chi(-1) = -1.\end{cases}$$

Define the *completed L-function* of χ as

$$\Lambda(1-s,\chi) := \frac{G(\chi)}{((-1)^{a_\chi}N)^{\frac{1}{2}}} L(s,\overline{\chi}).$$

The following holds:

$$\Lambda(s,\chi) = \left(\frac{N}{\pi}\right)^{\frac{s+a_\chi}{2}} \Gamma\left(\frac{s+a_\chi}{2}\right)\Lambda(s,\chi). \tag{1.5}$$

For a proof of this fact, the reader can consult [**10**, §3.4.3], for example.

1.2.1. Modular forms. Let $\mathfrak{H}$ denote the *upper half-plane*

$$\{z \in \mathbb{C} : \Im(z) > 0\}.$$

For every τ in $\mathfrak{H}$ and z in $\mathbb{C}$, write q for $e^{2\pi i\tau}$ and ζ for $e^{2\pi iz}$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathfrak{H}$ via linear fractional transformations:

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tau\right) \mapsto \begin{pmatrix} a & b \\ c & d\end{pmatrix}\tau := \frac{a\tau+b}{c\tau+d}.$$

For every $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $\mathrm{GL}_2^+(\mathbb{R})$ and every τ in $\mathfrak{H}$, define the automorphy factor $j(A,\tau) := c\tau + d$. For every integer k, define a right-action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the space of functions $f : \mathfrak{H} \to \mathbb{C}$ in the following way:

$$(f, A) \mapsto (f|_k A)(\tau) := \det(A)^{\frac{k}{2}} j(A,\tau)^{-k} f(A\tau).$$

Let Γ denote the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ and, for every positive integer N, set

$$\Gamma(N) := \left\{A \in \Gamma : A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \bmod N\right\} \text{ and}$$

$$\Gamma_0(N) := \left\{A \in \Gamma : A \equiv \begin{pmatrix} * & * \\ 0 & *\end{pmatrix} \bmod N\right\}.$$

A *congruence subgroup* of Γ is a subgroup containing $\Gamma(N)$ for some N. The smallest possible such N is called the *level* of the congruence subgroup. A *cusp* of a congruence subgroup G is an equivalence class of $\mathbb{P}^1(\mathbb{Q})$ under the action of G and a representative of such an equivalence class is also called a cusp. A *multiplier system* of weight k for G is a homomorphism $v : G \to S^1$ if $k \in \mathbb{Z}$, or a function $v : G \to S^1$ such that $v(g_1)v(g_2) = \sigma(g_1, g_2)v(g_1g_2)$ for every g_1, g_2 in G, where $\sigma(g_1, g_2) = j(g_1, g_2\tau)^{1/2} j(g_2,\tau)^{1/2} j(g_1g_2,\tau)^{-1/2} \in \{\pm 1\}$ is independent of τ, if $k \in \mathbb{Z} + \frac{1}{2}$. In addition, v must satisfy $v(-I_2) = e^{-\pi ik}$ if $-I_2 \in G$.

Let $k \in \mathbb{Z}$, $N \in \mathbb{N}$, let G be a congruence subgroup of level N and let v be a multiplier system of weight k for G. An *elliptic modular form of weight k with multiplier system v for G* is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which satisfies the following properties:

- $f|_k A = v(A)f$ for every A in G,
- the function f is holomorphic at the cusps of G.

Every f as above has a Fourier expansion of the form

$$f(\tau) = \sum_{n \geq 0} a_f(n) q^{n/w},$$

where w is the *width* of the cusp $i\infty$ [**10**, §7.1]. The elliptic modular form f is called a *cusp form* if it vanishes at the cusps of G. The $\mathbb{C}$-vector space of elliptic modular forms of weight k with trivial multiplier system for $\Gamma_0(N)$ is denoted by $M_k(N)$ and its subspace of cusp forms is denoted by $S_k(N)$. Let $M_*(N)$ denote the graded ring $\oplus_{k\in\mathbb{Z}} M_k(N)$. If χ is a Dirichlet character modulo N and $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$, then set

$\chi(A) := \chi(d)$. The map $A \mapsto \chi(A)$ defines a multiplier system of even integral weight for $\Gamma_0(N)$, which we denote by the same symbol χ. Denote the $\mathbb{C}$-vector space of elliptic modular forms of weight k with character χ for $\Gamma_0(N)$ by $M_k(N,\chi)$ and its subspace of cusp forms by $S_k(N,\chi)$.

It is also possible to define elliptic modular forms of *half-integral* weight, whose theory was established by Shimura [**36**]. For example, the Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\tfrac{12}{n}\right) q^{\frac{n^2}{24}} \tag{1.6}$$

is a modular form of weight $1/2$ for Γ with multiplier system

$$v_\eta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{cases} \left(\frac{d}{|c|}\right)\exp\left(\frac{\pi}{12}((a+d-3)c - bd(c^2-1))\right), & \text{if } 2 \nmid c, \\ \left(\frac{c}{|d|}\right)\exp\left(\frac{\pi}{12}((a-2d)c - bd(c^2-1)+3d-3)\right)\varepsilon(c,d), & \text{if } 2 \mid c, \end{cases}$$

where

$$\varepsilon(c,d) := \begin{cases} -1, & \text{if } c \leq 0 \text{ and } d < 0 \text{ and} \\ 1, & \text{otherwise.}\end{cases}$$

The multiplier system v_η is a projective character of order 24 [**10**, §5.8]. Together with the *scalar Jacobi theta series*, the Dedekind η-function can be used as a building block for Jacobi forms, as we shall see in Subsection 1.2.5.

For every l in $\mathbb{N}$, define the following operators on $M_k(N,\chi)$:

$$U(l)f(\tau) := \sum_{n\geq 0} a_f(ln)q^n,$$

$$V(l)f(\tau) := \sum_{n\geq 0} a_f(n)q^{ln},$$

$$T(l)f(\tau) := l^{\frac{k}{2}-1}\sum_{ad=l}\sum_{b \bmod d} \chi(a) f|_k \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)(\tau). \tag{1.7}$$

It is well-known that the Hecke operators $T(\cdot)$ map $M_k(N,\chi)$ to itself and that $U(l)$ and $V(l)$ map $M_k(N,\chi)$ to $M_k(lN,\chi)$ (see [**11**, §5], for example). Furthermore, if $l \mid N$, then $U(l)f$ is an element of $M_k(N,\chi)$.

Let $f = \sum_n a_f(n)q^n$ be an elliptic modular form in $M_k(N,\chi)$, which is a normalized eigenfunction of the Hecke operators $T(l)$ for all l in $\mathbb{N}$. The L-series of f in s is defined as

$$L(s,f) = \sum_{n=1}^{\infty} a_f(n)n^{-s}.$$

It has an Euler product of the form

$$L(s,f) = \prod_p \left(1 - a_f(p)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1}. \tag{1.8}$$

The reader can consult [**10**, §10.7] for a proof of this fact when $f \in M_k(1)$ and the same argument holds for f in $M_k(N,\chi)$. Define the *completed L-function* of f as

$$\Lambda_N(s,f) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)L(s,f).$$

Definition 1.3 (Metaplectic group). The metaplectic group, denoted by $\tilde{\Gamma}$, consists of pairs $\tilde{A} := (A, w(\tau))$ with A in Γ and $w : \mathfrak{H} \to \mathbb{C}$ a holomorphic function satisfying $w(\tau)^2 = j(A,\tau)$. The group law on $\tilde{\Gamma}$ is

$$(A, w(\tau))(B, v(\tau)) = (AB, w(B\tau)v(\tau)).$$

The metaplectic group is a double cover of Γ and it is generated by the following elements:

$$\tilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \qquad \text{and} \qquad \tilde{S} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

Definition 1.4 (Vector-valued modular forms). Let V be a finite-dimensional vector space over $\mathbb{C}$. For every half-integer k, define a right-action of $\tilde{\Gamma}$ on the space of functions $F : \mathfrak{H} \to V$ in the following way:

$$(F, \tilde{A}) \mapsto F|_k\tilde{A}(\tau) := w(\tau)^{-2k}F(A\tau).$$

Let $\rho : \tilde{\Gamma} \to \mathrm{Aut}(V)$ be a finite-dimensional representation of $\tilde{\Gamma}$, whose kernel has finite index in $\tilde{\Gamma}$. A vector-valued modular form of weight k for ρ is a holomorphic function $F : \mathfrak{H} \to V$ which satisfies

$$F|_k\tilde{A}(\tau) = \rho(\tilde{A})F(\tau)$$

for every element $\tilde{A}$ of $\tilde{\Gamma}$ and whose individual components F_j ($1 \leq j \leq \dim(V)$) extend to holomorphic functions from $\mathfrak{H}$ to $\mathbb{C}$. Denote the $\mathbb{C}$-vector space of all such functions by $M_k(\rho)$.

Let I_n denote the $n \times n$ identity matrix and set $E_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. The *symplectic group* $\mathrm{Sp}_n(\mathbb{R})$ is the set of $2n \times 2n$ matrices M in $\mathrm{GL}_n(\mathbb{R})$ satisfying $M^T E_n M = E_n$. We often consider its subgroup $\mathrm{Sp}_n(\mathbb{Z})$ of matrices with integer entries. The *Siegel upper half-space of degree n*, denoted by $\mathfrak{H}_n$, is the set of complex, symmetric $n \times n$ matrices with positive-definite imaginary part. The group $\mathrm{Sp}_n(\mathbb{R})$ acts on $\mathfrak{H}_n$ via

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} Z := (AZ + B)(CZ + D)^{-1}.$$

Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $n > 1$ and let G be a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$. A *Siegel modular form of weight k and degree n for G* is a holomorphic function $F : \mathfrak{H}_n \to \mathbb{C}$ which satisfies

$$F\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z\right) = \det(CZ + D)^k F(Z)$$

for every $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in G. An analogous definition can be given for every finite index subgroup of $\mathrm{Sp}_n(\mathbb{Z})$.

1.2.2. Lattices. Let R be a commutative ring and let L and N be R-modules, with L free of finite rank equal to g. A map $\beta : L \times L \to N$ is called a *symmetric R-bilinear form* if

$$\beta(x, y) = \beta(y, x) \qquad \text{and} \qquad \beta(x, my + nz) = m\beta(x, y) + n\beta(x, z)$$

for all x, y, z in L and all m, n in R. If $N = R$, then β is called *integral*. If $\beta(x, y) = 0$ for all y in L if and only if $x = 0$, then β is called *non-degenerate*. Let $\{e_1, \ldots, e_g\}$ be an R-basis of L. The matrix $G = (\beta(e_i, e_j))_{i,j}$ is called the *Gram matrix* of β with respect to $\{e_1, \ldots, e_g\}$. Let $\tilde{x}$ and $\tilde{y}$ be the column vectors whose entries are the coefficients of x and y with respect to $\{e_1, \ldots, e_g\}$. Then

$$\beta(x, y) = \sum_{i=1}^{g} \sum_{j=1}^{g} \tilde{x}_i \tilde{y}_j \beta(e_i, e_j) = \tilde{x}^t G \tilde{y}. \tag{1.9}$$

Definition 1.5 (Lattice). Let L and N be R-modules, with L free of finite rank, and let $\beta : L \times L \to N$ be a symmetric, non-degenerate bilinear form. The pair $\underline{L} = (L, \beta)$ is called a lattice over R.

The lattice $\underline{L}$ is called *integral* if the associated bilinear form is integral. By abuse of notation, denote the quadratic form associated with $\underline{L}$ by $\beta(\cdot)$, i.e. $\beta(x) := \frac{1}{2}\beta(x,x)$.

Throughout this thesis, we consider only $R = \mathbb{Z}$. Given an arbitrary $\mathbb{Z}$-basis of L, identify every element in the lattice with its coefficient vector and drop the tilde from the notation, i.e. write $\beta(x,y) = x^t G y$. Using the matrix formula (1.9), it is possible to extend the domain of definition of β to $L \otimes_{\mathbb{Z}} \mathbb{Q}$, $L \otimes_{\mathbb{Z}} \mathbb{R}$ and $L \otimes_{\mathbb{Z}} \mathbb{C}$ in a natural way. For every $z = (z_1, \ldots, z_{\mathrm{rk}(\underline{L})})$ in $L \otimes_{\mathbb{Z}} \mathbb{C}$, let $\mathfrak{R}(z) = (\mathfrak{R}(z_1), \ldots, \mathfrak{R}(z_{\mathrm{rk}(\underline{L})}))$ and $\mathfrak{I}(z) = (\mathfrak{I}(z_1), \ldots, \mathfrak{I}(z_{\mathrm{rk}(\underline{L})}))$ denote its real and imaginary parts, respectively.

An integral lattice $\underline{L} = (L, \beta)$ is called *positive-definite* if $\beta(x,x) > 0$ for all x in L such that $x \neq 0$. It is called *even* if $\beta(x,x)$ is even for all x in L, otherwise it is called *odd*. The *rank* of $\underline{L} = (L,\beta)$, denoted by $\mathrm{rk}(\underline{L})$, is defined as the rank of L as a $\mathbb{Z}$-module.

Example 1.6. The following are examples of positive-definite, even lattices over $\mathbb{Z}$:

(1) For every positive integer m, the lattice $\underline{L}_m := (\mathbb{Z}, (x,y) \mapsto 2mxy)$.
(2) More generally, for every positive-definite, even, $g \times g$ matrix M, the lattice $\underline{L}_G := (\mathbb{Z}^g, (x,y) \mapsto x^t G y)$.
(3) For every positive integer n, the $\mathbb{Z}$-module
$$D_n = \{(x_1, x_2, \ldots, x_n) \subseteq \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\},$$
equipped with the Euclidean bilinear form
$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) \mapsto x_1 y_1 + \cdots + x_n y_n.$$
(4) For every positive integer n, the $\mathbb{Z}$-module
$$A_n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} : x_1 + \cdots + x_{n+1} = 0\right\},$$
equipped with the Euclidean bilinear form.
(5) The $\mathbb{Z}$-module
$$E_8 = \left\{(x_1, x_2, \ldots, x_8) : \text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2}, x_1 + \cdots + x_8 \in 2\mathbb{Z}\right\},$$
equipped with the Euclidean bilinear form.

Definition 1.7. For every lattice $\underline{L} = (L,\beta)$ and every m in $\mathbb{Z}$, set $\underline{L}(m) := (L, m\beta)$.

If M is a free sub-module of L of finite rank equal to $\mathrm{rk}(\underline{L})$ and M has finite index in L, then (M,β) is called a *sublattice* of $\underline{L}$ and $\underline{L}$ is called an *overlattice* of (M,β). Two lattices $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ are *isomorphic* if there exists an isomorphism of underlying $\mathbb{Z}$-modules $\sigma : L_1 \xrightarrow{\sim} L_2$ such that $\beta_1 = \beta_2 \circ \sigma$. The isomorphisms between $\underline{L}$ and itself form the *orthogonal group of* $\underline{L}$, denoted by $O(\underline{L})$.

For the remainder of this section, assume that $\underline{L} = (L,\beta)$ is an even lattice over $\mathbb{Z}$. Define the following $\mathbb{Z}$-module:
$$L^\# = \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(y,x) \in \mathbb{Z} \ \forall\, x \text{ in } L\}.$$
The *dual lattice* of $\underline{L}$ is the pair $\underline{L}^\# = (L^\#, \beta)$. It is well-known that, if $\underline{L}$ has Gram matrix G with respect to some basis $\{e_1, \ldots, e_{\mathrm{rk}(\underline{L})}\}$ of L, then a $\mathbb{Z}$-basis of $L^\#$ is given by the *dual basis* $\{e_1^\#, \ldots, e_{\mathrm{rk}(\underline{L})}^\#\}$, where
$$e_i^\# = \sum_{j=1}^{\mathrm{rk}(\underline{L})} G_{ji}^{-1} e_j$$
and the Gram matrix of $\underline{L}^\#$ with respect to this basis is equal to G^{-1}.

An integral lattice $\underline{L} = (L,\beta)$ is called *unimodular* if $L^\# = L$. For example, the lattice E_8 from Example 1.6, (5) is unimodular.

If $\{f_1, \ldots, f_{\mathrm{rk}(\underline{L})}\}$ is another $\mathbb{Z}$-basis of L, then consider the change of coordinates map

$$U : L \to L, U(e_i) = \sum_{j=1}^{\mathrm{rk}(\underline{L})} U_{ji} f_j.$$

Its matrix $U = (U_{ij})_{i,j}$ is an element of $\mathrm{GL}_{\mathrm{rk}(\underline{L})}(\mathbb{Z})$ and, if $\overline{x}$ is the column vector whose entries are the coefficients of x with respect to the new basis, then $U\tilde{x} = \overline{x}$ and the Gram matrix of $\underline{L}$ with respect to $\{f_1, \ldots, f_{\mathrm{rk}(\underline{L})}\}$ is equal to $G' = (U^{-1})^t G U^{-1}$. Let G be the Gram matrix of $\underline{L}$ with respect to some basis of L. The *determinant* of $\underline{L}$ is defined as $\det(\underline{L}) := |\det(G)|$. The previous discussion implies that this quantity is independent of change of basis. It is well-known that $L^{\#}/L$ is a finite abelian group of order equal to $\det(\underline{L})$.

The *level* of $\underline{L}$, denoted by $\mathrm{lev}(\underline{L})$, is the smallest positive integer which satisfies $\mathrm{lev}(\underline{L})\beta(x) \in \mathbb{Z}$ for all x in $L^{\#}$. It is well-known that $\mathrm{lev}(\underline{L})$ is the smallest positive integer such that $\mathrm{lev}(\underline{L})G^{-1}$ is an even matrix, independent of the choice of basis for $\underline{L}$ [**13**, §3.1]. The following remark from [**10**, §14.3] plays an important role throughout this thesis:

Remark 1.8. If $\underline{L}$ is even, then $\mathrm{lev}(\underline{L})L^{\#} \subseteq L$. Furthermore, the level and the discriminant of $\underline{L}$ have the same set of prime divisors: if $\mathrm{rk}(\underline{L})$ is even, then $\mathrm{lev}(\underline{L}) \mid \det(\underline{L}) \mid \mathrm{lev}(\underline{L})^{\mathrm{rk}(\underline{L})}$ and, if $\mathrm{rk}(\underline{L})$ is odd, then $2 \mid \det(\underline{L})$ and $4 \mid \mathrm{lev}(\underline{L}) \mid 2\det(\underline{L}) \mid \mathrm{lev}(\underline{L})^{\mathrm{rk}(\underline{L})}$.

Definition 1.9. Set

$$\Delta(\underline{L}) := \begin{cases} (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L}), & \text{if } \mathrm{rk}(\underline{L}) \equiv 0 \bmod 2 \text{ and} \\ (-1)^{\lfloor \frac{\mathrm{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L}), & \text{if } \mathrm{rk}(\underline{L}) \equiv 1 \bmod 2. \end{cases}$$

It is well-known that $\Delta(\underline{L})$ is a discriminant (see Lemma 14.3.20 and Remark 14.3.23 in [**10**, §14.3]).

Definition 1.10. For every a in $\mathbb{N}$ and every D in $\mathbb{Q}$ such that $D\Delta(\underline{L}) \in \mathbb{Z}$, set

$$\chi_{\underline{L}}(D, a) := \left(\frac{D\cdot\Delta(\underline{L})}{a}\right) \qquad \text{and} \qquad \chi_{\underline{L}}(a) := \chi_{\underline{L}}(1, a).$$

Since $\Delta(\underline{L})$ is a discriminant, the function $\chi_{\underline{L}}(\cdot)$ is a well-defined quadratic character modulo $|\Delta(\underline{L})|$.

Definition 1.11 (Finite quadratic module). A finite quadratic module over $\mathbb{Z}$ is a pair (M, Q), such that M is an abelian group of finite order and $Q : M \to \mathbb{Q}/\mathbb{Z}$ is a non-degenerate quadratic form on M, i.e

- $Q(ax) = a^2 Q(x)$ for all a in $\mathbb{Z}$ and all x in M,
- the symmetric form $\beta : M \times M \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\beta(x, y) = Q(x + y) - Q(x) - Q(y)$$

is $\mathbb{Z}$-bilinear and non-degenerate.

The following result is [**41**, Theorem 1.1.8]:

Theorem 1.12. *Every finite quadratic module (M, Q) is isomorphic to a direct sum of finite quadratic modules of the following type (called Jordan constituents):*

- $A^t_{p^n} := (\mathbb{Z}_{(p^n)}, r \mapsto \frac{tr^2}{p^n})$, *for some odd prime p and some integer t such that $(t, p) = 1$,*
- $A^t_{2^n} := (\mathbb{Z}_{(2^n)}, r \mapsto \frac{tr^2}{2^{n+1}})$, *for some odd integer t,*
- $B_{2^n} := \left(\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}, (r, s) \mapsto \frac{r^2+rs+s^2}{2^n}\right)$,

- $C_{2^n} := \left(\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}, (r,s) \mapsto \frac{rs}{2^n}\right)$.

Definition 1.13 (Discriminant module). When $\underline{L}$ is even, the reduction of β modulo $\mathbb{Z}$ induces a bilinear form on $L^\#/L$. The discriminant module associated with the lattice $\underline{L}$ is the pair

$$D_{\underline{L}} := \left(L^\#/L, x + L \mapsto \beta(x) + \mathbb{Z}\right).$$

It is a finite quadratic module over $\mathbb{Z}$.

The orthogonal group of $D_{\underline{L}}$, denoted by $O(D_{\underline{L}})$, consists of all automorphisms α of $L^\#/L$ such that $\beta \circ \alpha = \beta$. Every automorphism of $\underline{L}$ extends to an automorphism of $\underline{L}^\#$, which in turn induces an automorphism of $D_{\underline{L}}$. Hence, there is an induced homomorphism between $O(\underline{L})$ and $O(D_{\underline{L}})$ (which need not be injective or surjective).

For every element x in $L^\#$, let N_x denote the order of $x + L$ in $L^\#/L$, i.e. the smallest positive integer such that $N_x x \in L$. Let $\operatorname{lev}(x)$ denote the smallest positive integer such that $\operatorname{lev}(x)\beta(x) \in \mathbb{Z}$.

Remark 1.14. Since $\beta(x, N_x x) \in \mathbb{Z}$ and $\beta(N_x x, N_x x) \in 2\mathbb{Z}$ for every x in $L^\#$, it follows that $\operatorname{lev}(x) \mid 2N_x$ and that $\operatorname{lev}(x) \mid N_x^2$. In particular, we have $\operatorname{lev}(x) \mid N_x$ when N_x is odd.

The *isotropy set* of $D_{\underline{L}}$ is

$$\operatorname{Iso}(D_{\underline{L}}) := \{x \in D_{\underline{L}} : \beta(x) = 0\}.$$

Lfet $\mathcal{I}_{\underline{L}}$ denote the set of isotropic subgroups of $D_{\underline{L}}$.

Definition 1.15. There is an action of $\mathbb{Z}^\times_{(\operatorname{lev}(\underline{L}))}$ on $\operatorname{Iso}(D_{\underline{L}})$ given by right multiplication. Let $\mathscr{R}_{\mathrm{Iso}}$ be a set of representatives of the orbit space $\operatorname{Iso}(D_{\underline{L}})/\mathbb{Z}^\times_{(\operatorname{lev}(\underline{L}))}$.

Consider the group algebra $\mathbb{C}[L^\#/L]$ of maps $L^\#/L \to \mathbb{C}$, with natural basis $\{\mathfrak{e}_x\}_{x\in L^\#/L}$. Define a scalar product on $\mathbb{C}[L^\#/L]$ as

$$\langle \sum_{x\in L^\#/L} f_x \mathfrak{e}_x, \sum_{x\in L^\#/L} g_x \mathfrak{e}_x \rangle := \sum_{x\in L^\#/L} f_x \overline{g_x}.$$

Definition 1.16 (Weil representation). Define the Weil representation associated with $\underline{L}$ of $\tilde{\Gamma}$ on $\operatorname{Aut}(\mathbb{C}[L^\#/L])$ by the following action of the generators of $\tilde{\Gamma}$ on the basis elements of $\mathbb{C}[L^\#/L]$:

$$\rho_{\underline{L}}(\tilde{T})\mathfrak{e}_x = e(\beta(x))\mathfrak{e}_x,$$

$$\rho_{\underline{L}}(\tilde{S})\mathfrak{e}_x = \frac{i^{-\frac{\operatorname{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}} \sum_{y\in L^\#/L} e(-\beta(x,y))\mathfrak{e}_y.$$

In general, write

$$\rho_{\underline{L}}(\tilde{A})\mathfrak{e}_y = \sum_{x\in L^\#/L} \rho_{\underline{L}}(\tilde{A})_{x,y}\mathfrak{e}_x$$

for every element $\tilde{A}$ of $\tilde{\Gamma}$. It is well-known that $\rho_{\underline{L}}$ is unitary and hence its dual representation is given by the formula

$$\rho^*_{\underline{L}}(\tilde{A})\mathfrak{e}_y = \sum_{x\in L^\#/L} \overline{\rho_{\underline{L}}(\tilde{A})_{x,y}}\mathfrak{e}_x.$$

Definition 1.17 (Direct sum of two lattices). Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be two even lattices and define a symmetric, non-degenerate bilinear form on $L_1 \oplus L_2$ as $f : (L_1 \oplus L_2) \times (L_1 \oplus L_2) \to \mathbb{Z}$,

$$f(x_1 \oplus x_2, y_1 \oplus y_2) := \beta_1(x_1, y_1) + \beta_2(x_2, y_2).$$

The direct sum of $\underline{L}_1$ and $\underline{L}_2$ is the even lattice $\underline{L}_1 \oplus \underline{L}_2 := (L_1 \oplus L_2, f)$.

Definition 1.18 (Stably isomorphic lattices). Two even lattices $\underline{L}_1$ and $\underline{L}_2$ are stably isomorphic if and only if there exist even unimodular lattices $\underline{U}_1$ and $\underline{U}_2$ such that $\underline{L}_1 \oplus \underline{U}_1 \simeq \underline{L}_2 \oplus \underline{U}_2$.

The following result was proved in [**28**, §1.3]:

Theorem 1.19. *Two even integral lattices are stably isomorphic if and only if their discriminant modules are isomorphic.*

Let F be a field of characteristic different from two. A *quadratic space* over F is a pair (V, Q), such that V is a finite-dimensional F-module and $Q : V \to F$ is a quadratic form on V. Let (V_1, Q_1) and (V_2, Q_2) be two quadratic spaces over F. A *representation of* V_1 *into* V_2 *with respect to* Q_1 *and* Q_2 is a linear map $\sigma : V_1 \to V_2$ which satisfies

$$Q_2 \circ \sigma(x) = Q_1(x), \quad \text{for all } x \text{ in } V_1.$$

When $F = \mathbb{Q}$, every such function can be extended to a function $\sigma : V_1 \otimes_{\mathbb{Z}} \mathbb{C} \to V_2 \otimes_{\mathbb{Z}} \mathbb{C}$ by linearity. If β_1 and β_2 denote the bilinear forms associated with Q_1 and Q_2, respectively, then every representation σ of V_1 into V_2 satisfies

$$\beta_2\left(\sigma(x), \sigma(y)\right) = \beta_1(x, y) \quad \text{for all } x, y \text{ in } V_1.$$

An *isometry* of (V_1, Q_1) into (V_2, Q_2) is an injective representation of V_1 into V_2 with respect to Q_1 and Q_2.

Definition 1.20 (Isometry of lattices). Let $\underline{L}_1$ and $\underline{L}_2$ be lattices in (V_1, Q_1) and (V_2, Q_2), respectively. An isometry of $\underline{L}_1$ into $\underline{L}_2$ is an isometry σ of (V_1, Q_1) into (V_2, Q_2), such that $\sigma\underline{L}_1 \subseteq \underline{L}_2$.

Fix any two bases $\mathbb{Z}$-bases of L_1 and L_2 and let G_1 and G_2 denote the Gram matrices of $\underline{L}_1$ and $\underline{L}_2$, respectively. Let M denote the matrix of σ with respect to these bases. The relation $Q_2 \circ \sigma = Q_1$ implies that

$$M^t G_2 M = G_1.$$

Hence, if T and U are change of coordinates maps for $\underline{L}_1$ and $\underline{L}_2$, respectively, then the matrix of σ with respect to the new bases is equal to UMT^{-1}. When $\mathrm{rk}(\underline{L}_1) = \mathrm{rk}(\underline{L}_2)$, set $\det(\sigma) := |\det(M)|$.

1.2.3. Jacobi modular forms. For the remainder of this chapter, assume that $\underline{L} = (L, \beta)$ is a positive-definite, even lattice over $\mathbb{Z}$. In order to define the Jacobi group, we first need to define the Heisenberg group. This group originates from quantum mechanics, more precisely in the description of one-dimensional mechanical systems. In number theory, it is intimately related to theta series via its *theta representation*. For details on this topic, see [**27**, §1.3]. We follow the exposition in [**1**] and the reader can consult the cited text for details and proofs.

Definition 1.21 (Heisenberg group). The Heisenberg group associated with $\underline{L}$ is the set

$$H^{\underline{L}}(\mathbb{R}) := \{(x, y, \zeta) : x, y \in L \otimes_{\mathbb{Z}} \mathbb{R}, \zeta \in S^1\},$$

equipped with the following composition law:

$$(x_1, y_1, \zeta_1)(x_2, y_2, \zeta_2) := (x_1 + x_2, y_1 + y_2, \zeta_1\zeta_2 e\left(\beta(x_1, y_2)\right)).$$

The *integral Heisenberg group* is the subgroup $H^{\underline{L}}(\mathbb{Z}) := \{(x, y, 1) : x, y \in L\}$ of $H^{\underline{L}}(\mathbb{R})$. Drop the third entry from the notation for this group for simplicity.

Proposition 1.22. *The group* $\mathrm{SL}_2(\mathbb{R})$ *acts on* $H^{\underline{L}}(\mathbb{R})$ *from the right via*

$$((x,y,\zeta),A) \mapsto (x,y,\zeta)^A := ((x,y)A, \zeta e_2(\beta((x,y)A) - \beta(x,y))),$$

where $(x,y)A$ *is the vector obtained by multiplying the row vector* (x,y) *with the matrix* A.

Definition 1.23 (Jacobi group). The real Jacobi group associated with $\underline{L}$, denoted by $J^{\underline{L}}(\mathbb{R})$, is the semi-direct product of $\mathrm{SL}_2(\mathbb{R})$ and $H^{\underline{L}}(\mathbb{R})$. The composition law on this group is

$$(A,h)\cdot(A',h') = (AA', h^{A'}h').$$

The following holds:

Proposition 1.24. *The real Jacobi group acts on the left on the space* $\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C})$. *If* $A \in \mathrm{SL}_2(\mathbb{R})$ *and* $h = (x,y,\zeta) \in H^{\underline{L}}(\mathbb{R})$, *then the action of* (A,h) *on a pair* (τ,z) *in* $\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C})$ *is defined as*

$$((A,h),(\tau,z)) \mapsto (A,h)(\tau,z) := \left(A\tau, \frac{z+x\tau+y}{j(A,\tau)}\right).$$

The real Jacobi group also acts on the space of holomorphic, complex-valued functions defined on $\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C})$.

Definition 1.25 (Jacobi slash operator). Let k be a positive integer and let $\phi : \mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C}) \to \mathbb{C}$ be a holomorphic function. For every $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$, set

$$\phi|_{k,\underline{L}}A(\tau,z) := \phi\left(A\tau, \frac{z}{j(A,\tau)}\right) j(A,\tau)^{-k} e\left(\frac{-c\beta(z)}{j(A,\tau)}\right)$$

and, for every $h = (x,y,\zeta)$ in $H^{\underline{L}}(\mathbb{R})$, set

$$\phi|_{\underline{L}}h(\tau,z) := \zeta\phi(\tau, z+x\tau+y)e(\tau\beta(x)+\beta(x,z)).$$

The action of $J^{\underline{L}}(\mathbb{R})$ on the space of holomorphic, complex-valued functions defined on $\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C})$ is defined as

$$(\phi,(A,h)) \mapsto \phi|_{k,\underline{L}}(A,h) := (\phi|_{k,\underline{L}}A)|_{\underline{L}}h. \tag{1.10}$$

Note that the actions of Γ and $H^{\underline{L}}(\mathbb{R})$ do not commute.

The *integral Jacobi group* is the subgroup $J^{\underline{L}}(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) \ltimes H^{\underline{L}}(\mathbb{Z})$ of $J^{\underline{L}}(\mathbb{R})$. From now on, drop the word "integral" from the language and the $(\mathbb{Z})$ from the notation for this group.

Definition 1.26 (Jacobi form of lattice index). Let k be a positive integer. A Jacobi form of weight k and index $\underline{L}$ is a holomorphic function $\phi : \mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C}) \to \mathbb{C}$ with the following properties:

(1) for all γ in $J^{\underline{L}}$, the following holds:

$$\phi|_{k,\underline{L}}\gamma(\tau,z) = \phi(\tau,z);$$

(2) the function ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^\# \\ n\geq\beta(r)}} c_\phi(n,r)e(n\tau+\beta(r,z)). \tag{1.11}$$

The complex numbers $c_\phi(\cdot,\cdot)$ are called the Fourier coefficients of ϕ.

For fixed weight and index, denote the $\mathbb{C}$-vector space of all such functions by $J_{k,\underline{L}}$.

REMARK 1.27. Consider the lattice $\underline{L}_G$ from Example 1.6, (2); then $J_{k,\underline{L}_G}$ is the space $J_{k,\frac{1}{2}G}$ of Jacobi forms of weight k and matrix index $\frac{1}{2}G$ defined in [**2**]. Consider the lattice $\underline{L}_m$ from Example 1.6, (1); then the space $J_{k,\underline{L}_m}$ is the space $J_{k,m}$ of Jacobi forms of weight k and scalar index m defined in [**14**].

It is also possible to define Jacobi forms of *half-integral* weight, of *odd* lattice index or with multiplier system. We do not go into further details and instead refer the reader to [**22**, §III.9]. The following lemma is a particular case of [**9**, Proposition 2.5]:

LEMMA 1.28. *Let $k_1, k_2 \in \mathbb{N}$ and let $\underline{L}_1$ and $\underline{L}_2$ be two positive-definite, even lattices over $\mathbb{Z}$. If $\phi(\tau, z_1) \in J_{k_1,\underline{L}_1}$ and $\psi(\tau, z_2) \in J_{k_2,\underline{L}_2}$, then*

$$\phi(\tau, z_1)\psi(\tau, z_2) \in J_{k_1+k_2,\underline{L}_1\oplus\underline{L}_2}.$$

We include the proof, since it is not given explicitly in [**9**]:

PROOF. We remind the reader of Definition 1.17 of the direct sum of two lattices. Set $\underline{L}_1 \oplus \underline{L}_2 = (L,\beta)$, $z := z_1 \oplus z_2$ and $\delta(\tau, z) := \phi(\tau, z_1)\psi(\tau, z_2)$ for simplicity. Then δ is a holomorphic, complex-valued function defined on $\mathfrak{H} \times ((L_1 \oplus L_2) \otimes_{\mathbb{Z}} \mathbb{C})$. For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, we have

$$\begin{aligned}
\delta(\tau, z)|_{k_1+k_2,\underline{L}_1\oplus\underline{L}_2}A(\tau, z) =&\delta\left(A\tau, \frac{z}{j(A,\tau)}\right) j(A,\tau)^{-(k_1+k_2)} e\left(\frac{-c\beta(z)}{j(A,\tau)}\right)\\
=&\phi\left(A\tau, \frac{z_1}{j(A,\tau)}\right)\psi\left(A\tau, \frac{z_2}{j(A,\tau)}\right) j(A,\tau)^{-k_1}\\
&\times j(A,\tau)^{-k_2} e\left(\frac{-c\beta_1(z_1)}{j(A,\tau)}\right) e\left(\frac{-c\beta_2(z_2)}{j(A,\tau)}\right)\\
=&\phi|_{k_1,\underline{L}_1}A(\tau, z_1)\psi|_{k_2,\underline{L}_2}A(\tau, z_2) = \delta(\tau, z),
\end{aligned}$$

since $\phi \in J_{k_1,\underline{L}_1}$ and $\psi \in J_{k_2,\underline{L}_2}$.

Each x in $H^{\underline{L}_1\oplus\underline{L}_2}(\mathbb{Z})$ can be written as $x = x_1 \oplus x_2$, with $x_1 \in H^{\underline{L}_1}(\mathbb{Z})$ and $x_2 \in H^{\underline{L}_2}(\mathbb{Z})$. For every $(\lambda, \mu) = (\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2)$ in $H^{\underline{L}_1\oplus\underline{L}_2}(\mathbb{Z})$, we have

$$\begin{aligned}
\delta|_{\underline{L}_1\oplus\underline{L}_2}(\lambda, \mu)(\tau, z) =&\delta(\tau, (z_1 + \lambda_1\tau + \mu_1) \oplus (z_2 + \lambda_2\tau + \mu_2))\\
&\times e(\tau(\beta_1(\lambda_1) + \beta_2(\lambda_2)) + \beta_1(\lambda_1, z_1) + \beta_2(\lambda_2, z_2))\\
=&\phi|_{\underline{L}_1}(\lambda_1, \mu_1)(\tau, z_1)\psi|_{\underline{L}_2}(\lambda_2, \mu_2)(\tau, z_2) = \phi(\tau, z_1)\psi(\tau, z_2)\\
=&\delta(\tau, z).
\end{aligned}$$

Suppose that ϕ and ψ have the following Fourier expansions:

$$\begin{aligned}
\phi(\tau, z_1) =& \sum_{\substack{n\in\mathbb{Z}, r\in L_1^\# \\ n\geq\beta_1(r)}} c_\phi(n, r) e\left(n\tau + \beta_1(r, z_1)\right) \text{ and}\\
\psi(\tau, z_2) =& \sum_{\substack{m\in\mathbb{Z}, s\in L_2^\# \\ m\geq\beta_2(s)}} c_\psi(m, s) e\left(m\tau + \beta_2(s, z_2)\right).
\end{aligned}$$

The $\mathbb{Z}$-module $L^\#$ contains elements of the form $y = y_1 \oplus y_2$ in $L \otimes_{\mathbb{Z}} \mathbb{Q}$, which satisfy $\beta(x, y) \in \mathbb{Z}$ for all $x = x_1 \oplus x_2$ in L. Take $x_1 = 0$ and then $x_2 = 0$ in order to obtain that $y_1 \in L_1^\#$ and $y_2 \in L_2^\#$, respectively. It follows that $L^\# \subseteq L_1^\# \oplus L_2^\#$ and the reverse inclusion also holds. Thus, we have $L^\# = L_1^\# \oplus L_2^\#$ and therefore

$$\begin{aligned}\delta(\tau,z) &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{\substack{r\in L_1^\#\\ n\geq\beta_1(r)}}\sum_{\substack{s\in L_2^\#\\ m\geq\beta_2(s)}} c_\phi(n,r)c_\psi(m,s)e\left((n+m)\tau+\beta_1(r,z_1)+\beta_2(s,z_2)\right)\\ &= \sum_{h\in\mathbb{Z}}\sum_{\substack{t\in L^\#\\ h\geq\beta(t)}}\Big(\sum_{n+m=h} c_\phi(n,r)c_\psi(m,s)\Big)e(h\tau+\beta(t,z)),\end{aligned}$$

where we have substituted $t = r\oplus s$. The sum over n and m has finite support and setting $f(h,t) := \sum_{n+m=h} c_\phi(n,r)c_\psi(m,s)$ leads to the desired form for the Fourier expansion of δ, completing the proof. □

The result in Lemma 1.28 can be extended inductively to the product of arbitrarily finitely many Jacobi forms $\phi_1,\dots,\phi_n$ of arbitrary weights and indexes. The following useful result is [**1**, Proposition 2.4.3]:

Proposition 1.29. *If ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the form*

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z},r\in L^\#\\ n\geq\beta(r)}} c_\phi(n,r)e\left(n\tau+\beta(r,z)\right),$$

then $c_\phi(n,r)$ depends only on $n-\beta(r)$ and on r mod L. More precisely, we have $c_\phi(n,r) = c_\phi(n',r')$ whenever $r\equiv r'$ mod L and $n-\beta(r) = n'-\beta(r')$.

Define the following set, called the support of $\underline{L}$:

$$\mathrm{supp}(\underline{L}) := \{(D,r) : D\in\mathbb{Q}_{\leq 0}, r\in L^\#, D\equiv\beta(r) \bmod \mathbb{Z}\}. \tag{1.12}$$

For every ϕ in $J_{k,\underline{L}}$ with Fourier expansion (1.11) and for each pair (D,r) in $\mathrm{supp}(\underline{L})$, set $C_\phi(D,r) := c_\phi(\beta(r)-D,r)$. Proposition 1.29 implies that every ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_\phi(D,r)e\left((\beta(r)-D)\tau+\beta(r,z)\right). \tag{1.13}$$

We will often use the interplay between these two Fourier expansions. In particular, use the latter to define cusp forms:

Definition 1.30 (Cusp form). A Jacobi form ϕ is called a cusp form if $C_\phi(0,r) = 0$ for all r in $L^\#$ such that $\beta(r)\in\mathbb{Z}$. Denote the $\mathbb{C}$-vector subspace of cusp forms in $J_{k,\underline{L}}$ by $S_{k,\underline{L}}$.

Definition 1.31 (Singular term). For each ϕ in $J_{k,\underline{L}}$, define its singular term as the series

$$C_0(\phi)(\tau,z) := \sum_{\substack{r\in L^\#\\ \beta(r)\in\mathbb{Z}}} C_\phi(0,r)e\left(\tau\beta(r)+\beta(r,z)\right).$$

Definition 1.32. Let r in $L^\#$ be such that $\beta(r)\in\mathbb{Z}$ and define the function

$$g_{\underline{L},r}(\tau,z) := e\left(\tau\beta(r)+\beta(r,z)\right)$$

on the space $\mathfrak{H}\times(L\otimes_\mathbb{Z}\mathbb{C})$.

Definition 1.33. Set

$$J_\infty^{\underline{L}} := \left\{\left(\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),(0,\mu)\right) : n\in\mathbb{Z},\mu\in L\right\}.$$

We will show in Chapter 2 that $J_\infty^{\underline{L}}$ is the stabilizer of the exponential functions $g_{\underline{L},r}(\cdot,\cdot)$ in $J^{\underline{L}}$. Jacobi–Eisenstein series are defined in the following way:

Definition 1.34 (Jacobi–Eisenstein series). Let k be a positive integer such that $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$. For each r in $\mathrm{Iso}(D_{\underline{L}})$, define the Jacobi–Eisenstein series of weight k and index $\underline{L}$ associated with r as

$$E_{k,\underline{L},r} := \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}}\gamma. \tag{1.14}$$

Define the subspace $J_{k,\underline{L}}^{\mathrm{Eis}}$ of $J_{k,\underline{L}}$ as the $\mathbb{C}$-span of the set $\{E_{k,\underline{L},r} : r \in \mathrm{Iso}(D_{\underline{L}})\}$. The series (1.14) converges under the imposed weight restrictions. It is possible to define Jacobi–Eisenstein series for $1 \le k \le \frac{\mathrm{rk}(\underline{L})}{2} + 2$ by using "Hecke's convergence trick", however we do not pursue this further. It was shown in [**1**, §3.3] that

$$E_{k,\underline{L},r} = (-1)^k E_{k,\underline{L},-r}. \tag{1.15}$$

It is also possible to define "twisted" Eisenstein series: for every r in $\mathrm{Iso}(D_{\underline{L}})$ and every Dirichlet character ξ modulo N_r, set

$$E_{k,\underline{L},r,\xi} := \sum_{d \in \mathbb{Z}_{(N_r)}} \xi(d) E_{k,\underline{L},dr}. \tag{1.16}$$

Since $\xi(d) = 0$ whenever $(d, N_r) > 1$ by definition, the above can be written as

$$E_{k,\underline{L},r,\xi} = \sum_{d \in \mathbb{Z}_{(N_r)}^\times} \xi(d) E_{k,\underline{L},dr}.$$

Equation (1.15) implies that

$$\begin{aligned} E_{k,\underline{L},r,\xi} =& (-1)^k \sum_{d \in \mathbb{Z}_{(N_r)}^\times} \xi(d) E_{k,\underline{L},-dr} = (-1)^k \sum_{d \in \mathbb{Z}_{(N_r)}^\times} \xi(-d) E_{k,\underline{L},dr} \\ =& (-1)^k \xi(-1) E_{k,\underline{L},r,\xi} \end{aligned}$$

and it follows that $E_{k,\underline{L},r,\xi}$ vanishes unless $\xi(-1) = (-1)^k$. We remind the reader of Definition 1.15 of the set $\mathscr{R}_{\mathrm{Iso}}$. If $x = r$ in $\mathscr{R}_{\mathrm{Iso}}$ (i.e. $x = er$ for some e in $\mathbb{Z}_{(\mathrm{lev}(\underline{L}))}^\times$), then

$$E_{k,\underline{L},x,\xi} = \sum_{d \in \mathbb{Z}_{(N_r)}^\times} \xi(d) E_{k,\underline{L},der} = \overline{\xi}(e) \sum_{f \in \mathbb{Z}_{(N_r)}^\times} \xi(f) E_{k,\underline{L},fr} = \overline{\xi}(e) E_{k,\underline{L},r,\xi}.$$

We have made the substitution $f = de$ and we have used Remark 1.8, which implies that multiplication by e is an isomorphism of $\mathbb{Z}_{(N_r)}^\times$. We have also used the fact that $N_x = N_r$ if $x = r$ in $\mathscr{R}_{\mathrm{Iso}}$, which we will prove in Lemma 3.52. By summing over all Dirichlet characters modulo N_r and using the character orthogonality relation

$$\sum_{\xi \bmod N_r} \xi(d)\overline{\xi(e)} = \begin{cases} |\mathbb{Z}_{(N_r)}^\times|, & \text{if } d = e \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

we obtain that

$$E_{k,\underline{L},r} = \frac{1}{|\mathbb{Z}_{(N_r)}^\times|} \sum_{\xi \bmod N_r} E_{k,\underline{L},r,\xi}.$$

In other words, the untwisted Eisenstein series can be recovered from the twisted ones.

It is well-known that, for every positive integer N, there exists a canonical bijection between the set of Dirichlet characters modulo N and the set of primitive Dirichlet characters whose conductor divides N. This bijection maps each Dirichlet character modulo N to the unique Dirichlet character which induces it (see Definition 1.2). Hence, we arrive to the following definition from [**1**]:

Definition 1.35. Let k in $\mathbb{N}$ be such that $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$ and let $r \in \mathscr{R}_{\mathrm{Iso}}$. For each primitive Dirichlet character χ modulo F with $F \mid N_r$ and $\chi(-1) = (-1)^k$, define

$$E_{k,\underline{L},r,\chi} := \sum_{d \in \mathbb{Z}_{(N_r)}^{\times}} \chi(d) E_{k,\underline{L},dr}.$$

For $k \leq \mathrm{rk}(\underline{L}) + 2$, the character χ has to be non-principal (i.e. $F \neq 1$) for convergence reasons.

For each Dirichlet character χ of modulus F as above, define a Dirichlet character $\tilde{\chi}$ of modulus N_r in the following way:

$$\tilde{\chi}(d) = \begin{cases} \chi(d), & \text{if } (d, N_r) = 1 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $E_{k,\underline{L},r,\chi} = E_{k,\underline{L},r,\tilde{\chi}}$, where the latter series is defined in (1.16).

For every x in $L^{\#}/L$, define the *Jacobi theta series*

$$\vartheta_{\underline{L},x}(\tau, z) := \sum_{\substack{r \in L^{\#} \\ r \equiv x \bmod L}} e\left(\tau\beta(r) + \beta(r, z)\right) \tag{1.17}$$

and set

$$\Theta_{\underline{L}} := \mathrm{Span}_{\mathbb{C}}\{\vartheta_{\underline{L},x} : x \in L^{\#}/L\}. \tag{1.18}$$

It was shown in [**3**, §3.5] that the series $\vartheta_{\underline{L},x}(\tau, \cdot)$ $(x \in L^{\#}/L)$ are linearly independent as functions of z. These functions are interesting in their own right and much can be said about them. We focus on their modular properties and refer the reader to [**3**, §3 and §4] for an in-depth discussion. Extend the definition of the $|_{k,\underline{L}}$-action of Γ on holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ to $\tilde{\Gamma}$ in the following way: for every k in $\frac{1}{2}\mathbb{Z}$ and every $\tilde{A} = (A, w(\tau))$ in $\tilde{\Gamma}$, set

$$\phi|_{k,\underline{L}}\tilde{A}(\tau, z) := \phi\left(A\tau, \frac{z}{w(\tau)^2}\right) w(\tau)^{-2k} e\left(\frac{-c\beta(z)}{w(\tau)^2}\right).$$

It was proved in [**3**, §3.5] that, for every $x \in L^{\#}/L$ and every $\tilde{A}$ as above, the theta series $\vartheta_{\underline{L},x}$ satisfies the following:

$$\vartheta_{\underline{L},x}|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\tilde{A} = \sum_{y \in L^{\#}/L} \rho_{\underline{L}}(\tilde{A})_{x,y}\vartheta_{\underline{L},y}. \tag{1.19}$$

In particular, the set $\Theta_{\underline{L}}$ is a $\tilde{\Gamma}$-module. For each ϕ in $J_{k,\underline{L}}$ with Fourier expansion (1.13), define the following function on the upper half-plane:

$$h_{\phi,x}(\tau) = \sum_{\substack{D \in \mathbb{Q} \\ (D,x) \in \mathrm{supp}(\underline{L})}} C_{\phi}(D, x) q^{-D}.$$

We will review the modular properties of $h_{\phi,x}$ in Subsection 1.2.4.3. It was shown in [**1**, §2.4] that every Jacobi form has a *theta expansion*:

Proposition 1.36. *Every Jacobi form ϕ in $J_{k,\underline{L}}$ can be written as*

$$\phi(\tau, z) = \sum_{x \in L^{\#}/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau, z). \tag{1.20}$$

The following result was proved in [**4**]:

THEOREM 1.37. *Let $\underline{L}_1$ and $\underline{L}_2$ be two positive-definite, even lattices over $\mathbb{Z}$ and assume that $j : D_{\underline{L}_1} \xrightarrow{\sim} D_{\underline{L}_2}$ is an isomorphism of finite quadratic modules. Then the map*

$$I_j : J_{k+\lceil \frac{rk(L_2)}{2} \rceil, \underline{L}_2} \to J_{k+\lceil \frac{rk(L_1)}{2} \rceil, \underline{L}_1}$$

defined by

$$\sum_{x \in L_2^\#/L_2} h_{\phi,x}(\tau)\vartheta_{\underline{L}_2,x}(\tau,z) \mapsto \sum_{x \in L_2^\#/L_2} h_{\phi,x}(\tau)\vartheta_{\underline{L}_1,j^{-1}(x)}(\tau,z)$$

is an isomorphism.

It was shown in [**1**, §3.3] that Eisenstein series can be written in terms of theta series as

$$E_{k,\underline{L},r} = \frac{1}{2} \sum_{A \in \Gamma_\infty \backslash \Gamma} \vartheta_{\underline{L},r}|_{k,\underline{L}} A. \tag{1.21}$$

In particular, the series $E_{k,\underline{L},r}$ only depends on r modulo L. Call $E_{k,\underline{L},0}$ the *trivial* Eisenstein series.

Next, define a scalar product on $S_{k,\underline{L}}$. For every τ in $\mathfrak{H}$ and z in $L \otimes_{\mathbb{Z}} \mathbb{C}$, let $\tau = u + iv$ and $z = x + iy$ be their decompositions into real and imaginary parts. In [**1**, §3.2], the author defines a $J^{\underline{L}}(\mathbb{R})$-invariant *volume element* on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ in the following way:

$$dV_{\underline{L},(\tau,z)} := v^{-\mathrm{rk}(\underline{L})-2} dudvdxdy.$$

For every pair of functions ϕ and ψ which are invariant under the $|_{k,\underline{L}}$-action of a subgroup Λ of $J^{\underline{L}}$ of finite index, set

$$\omega_{\phi,\psi}(\tau,z) := \phi(\tau,z)\overline{\psi(\tau,z)} v^k e^{-4\pi\beta(y)v^{-1}}. \tag{1.22}$$

It is easy to check that $\omega_{\phi,\psi}$ is also Λ-invariant.

DEFINITION 1.38 (Petersson scalar product). Let Λ be a subgroup of $J^{\underline{L}}$ of finite index and let $\mathfrak{F}_\Lambda$ denote a fundamental domain for the action of Λ on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$. If ϕ and ψ are two functions which are invariant under the $|_{k,\underline{L}}$-action of Λ and either one of them is a cusp form, define

$$\langle \phi, \psi \rangle_\Lambda := \frac{1}{[J^{\underline{L}} : \Lambda]} \int_{\mathfrak{F}_\Lambda} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)}. \tag{1.23}$$

The Petersson scalar product of two Jacobi forms does not depend on the choice of fundamental domain, or in fact of the subgroup Λ. Thus, drop the subscript from the notation and write $\langle \phi, \psi \rangle := \langle \phi, \psi \rangle_\Lambda$. Given a fundamental domain $\mathfrak{F}$ for the action of Γ on $\mathfrak{H}$ and a fundamental parallelotope $\mathfrak{P}$ for $(L \otimes_{\mathbb{Z}} \mathbb{C})/(\tau L + L)$, choose as a fundamental domain for the action of $J^{\underline{L}}$ on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ the set

$$\mathfrak{F}_{J^{\underline{L}}} := \{(\tau,z) \in \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) : \tau \in \mathfrak{F}, z \in \mathfrak{P}\}/\{\mathrm{id}, \iota\},$$

where ι is the reflection map $(\tau, z) \mapsto (\tau, -z)$.

According to [**1**, Proposition 3.2.10], the Petersson scalar product can be expressed in terms of theta expansions in the following way:

PROPOSITION 1.39. *Let*

$$\phi = \sum_{x \in L^\#/L} h_{\phi,x}\vartheta_{\underline{L},x} \qquad \text{and} \qquad \psi = \sum_{x \in L^\#/L} h_{\psi,x}\vartheta_{\underline{L},x}$$

be two Jacobi forms in $J_{k,\underline{L}}$ such that either one of them is a cusp form. Then

$$\langle \phi, \psi \rangle = 2^{-\frac{\mathrm{rk}(L)}{2}} \det(\underline{L})^{-\frac{1}{2}} \int_{\Gamma\backslash\mathfrak{H}} \sum_{x \in L^\#/L} h_{\phi,x}(\tau)\overline{h_{\psi,x}(\tau)} v^{k-\frac{\mathrm{rk}(L)}{2}-2} dudv.$$

In the proof of this Proposition given in [**1**], a scalar product is defined on $\Theta_{\underline{L}}$ by fixing a fiber τ in $\mathfrak{H}$ in (1.23):

$$\langle \sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}\rangle := \int_{\mathfrak{P}} \sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}(\tau,z) \sum_{s\in L^\#/L} \overline{d_s\vartheta_{\underline{L},s}(\tau,z)} v^{k-\mathrm{rk}(\underline{L})-2} e^{-4\pi\beta(y)v^{-1}}\,dxdy.$$

It was shown in [**1**, §3.2] that

$$\langle \sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}\rangle = v^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2}(2\det(\underline{L}))^{-\frac{\mathrm{rk}(\underline{L})}{2}} \sum_{r\in L^\#/L} c_r\overline{d_r}.$$

Let $[\cdot,\cdot]$ denote the following normalization of the above scalar product on $\Theta_{\underline{L}}$:

$$[\sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}] := \sum_{r\in L^\#/L} c_r\overline{d_r}. \tag{1.24}$$

This scalar product is non-degenerate.

1.2.4. Relations with other types of modular forms. In this subsection, we study the relation between Jacobi forms and Siegel modular forms and that between Jacobi forms and vector-valued modular forms for the dual Weil representation. These types of modular forms have been studied extensively in the literature and they represent crucial tools in the study of Jacobi forms. We also study the relation between Jacobi forms and orthogonal modular forms. Since the latter are not as well-known as other types of automorphic forms, we will give a brief overview.

Not a good reason to include in your thesis. I would prefer an overview by a specialist. E.g. Borcherds' papers give an excellent overview, as well as papers by Gritsenko.

1.2.4.1. *Jacobi forms of scalar index.* It is useful to have a background knowledge of the theory of Jacobi forms of scalar index. The integral *scalar Jacobi group* is $\Gamma^J := \Gamma \ltimes \mathbb{Z}^2$. This group acts on the right on the space of holomorphic, complex-valued functions defined on $\mathfrak{H}\times\mathbb{C}$. Let k and m be positive integers. For every $\gamma = (A,h)$ with $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in Γ and $h = (x,y)$ in $\mathbb{Z}^2$, set

$$\phi|_{k,m}\gamma(\tau,z) := \phi\left(A\tau, \frac{z+x\tau+y}{c\tau+d}\right)(c\tau+d)^{-k}e^m\left(\frac{-c(z+x\tau+y)^2}{c\tau+d} + x^2\tau + 2xz + xy\right).$$

This action agrees with Definition 1.25 when $\underline{L} = \underline{L}_m$ (see Remark 1.27). The space $J_{k,m}$ of *Jacobi forms of weight k and scalar index m* consists of all holomorphic functions $\phi : \mathfrak{H}\times\mathbb{C}\to\mathbb{C}$ with the following properties:

(1) for all (A,h) in Γ^J, we have $\phi|_{k,m}(A,h) = \phi$;
(2) the function ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n,r'\in\mathbb{Z}\\ 4mn-r'^2\geq 0}} b_\phi(n,r')e(n\tau + r'z). \tag{1.25}$$

Note that $L_m^\# = \frac{1}{2m}\mathbb{Z}$, $\det(\underline{L}_m) = 2m$ and $\mathrm{lev}(\underline{L}_m) = 4m$. Substitute $2mr = r'$ in (1.11) and set $b_\phi(n,r') := c_\phi(n,\frac{r'}{2m})$, in order to obtain the same expression as above. A scalar Jacobi form is called a *cusp form* if $b_\phi(n,r') = 0$ whenever $4mn = r^2$.

Example 1.40. Let $k \geq 4$ be an even integer. The Fourier expansion of the Eisenstein series $E_{k,\underline{L}_m,0}$ is computed in [**14**, §I.2]:

$$E_{k,\underline{L}_m,0}(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in\frac{1}{2m}\mathbb{Z}\\ n\geq mr^2}} e_{k,m}(n,r)e(\tau n + 2mrz); \tag{1.26}$$

if $n = mr^2$, then $e_{k,m}(n,r) = 1$ if $r \in \mathbb{Z}$ and it is equal to zero otherwise; if $n > mr^2$, then

$$e_{k,m}(n,r) = \frac{(-1)^{\frac{k}{2}}\pi^{k-\frac{1}{2}}}{m^{k-1}2^{k-2}\Gamma(k-\frac{1}{2})}(4nm - 4m^2r^2)^{k-\frac{3}{2}} \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{\lambda,d \bmod c \\ (d,c)=1}} e_c(md^{-1}\lambda^2 - 2mr\lambda + nd),$$

where d^{-1} denotes the inverse of d modulo c. Note that we have made the substitution $r = 2ms$ and relabelled $s = r$ in [**14**, §I.2, (5)]. When $m = 1$, the above expression simplifies to

$$e_{k,1}(n,r) = \frac{L_{4(r^2-n)}(2-k)}{\zeta(3-2k)},$$

where we remind the reader that $L_D(s) := L(s,\chi_D)$ for every discriminant D. When m is square-free,

$$e_{k,m}(n,r) = \frac{1}{\zeta(3-2k)\sigma_{k-1}(m)} \sum_{d|(n,2mr,m)} d^{k-1} L_{\frac{4m(mr^2-n)}{d^2}}(2-k)$$

and it is possible to obtain a similar expression for arbitrary m. We generalize these results in Section 2.3.

In general, write $m = ab^2$, where a is the square-free part of m, and define

$$E_{k,m,s}(\tau,z) := \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}_m} \backslash \Gamma^J} q^{as^2}\zeta^{2abs}|_{k,m}\gamma.$$

Then

$$q^{as^2}\zeta^{2abs} = e\left(m\tau\left(\frac{s}{b}\right)^2 + 2mz\frac{s}{b}\right) = g_{\underline{L},\frac{s}{b}}(\tau,z)$$

and the following holds:

$$\left\{\frac{s}{b} : s \in \mathbb{Z}_{(b)}\right\} = \mathrm{Iso}(D_{\underline{L}_m}).$$

To check that this is true, if $\frac{r}{2m} \in L^\#/L$, then $\beta(\frac{r}{2m}) = \frac{r^2}{4m}$ is an integer if and only if $4m \mid r^2$, i.e if and only if $4ab^2 \mid r^2$. This is equivalent to the condition that $r = 2abs$ for some s in $\mathbb{Z}$. On the other hand hand, it is clear that $\left\{\frac{s}{b} : s \in \mathbb{Z}_{(b)}\right\} \subseteq \mathrm{Iso}(D_{\underline{L}_m})$. It follows that $E_{k,m,s} = E_{k,\underline{L}_m,\frac{s}{b}}$. Twisted scalar Eisenstein series are defined in [**38**, §2] in the following way: for every divisor t of b and every primitive Dirichlet character χ modulo F with $F \mid \frac{b}{t}$ and $\chi(-1) = (-1)^k$, set

$$E_{k,m,t,\chi} := \sum_{d \bmod \frac{b}{t}} \chi(d)E_{k,m,td}.$$

The order of $\frac{t}{b}$ in $L_m^\#/L_m$ is equal to b/t and therefore

$$E_{k,m,t,\chi} = \sum_{d \bmod N_{\frac{t}{b}}} \chi(d)E_{k,\underline{L},\frac{td}{b}}.$$

This does not agree with Definition 1.35, since the corpimality conditions are missing in the summation.

EXAMPLE 1.41. It is also possible to define scalar Jacobi forms of *half-integral* weight and *half-integral* index. An important example is the scalar Jacobi theta series

$$\vartheta(\tau,z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) e\left(\tau\frac{n^2}{8} + \frac{nz}{2}\right), \tag{1.27}$$

which has weight $\frac{1}{2}$, index $\frac{1}{2}$ and multiplier system

$$v_\vartheta(A,(x,y)) = v_\eta(A)^3 \cdot (-1)^{x+y}.$$

It can be used as a building block for Jacobi forms, together with the Dedekind η-function.

just omit this remark.

Remark 1.42. The Jacobi group Γ^J can be embedded into $\mathrm{Sp}_2(\mathbb{Z})$ as the parabolic subgroup

$$\left\{\begin{pmatrix} * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z})\right\}, \tag{1.28}$$

via the map

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x,y)\right) \mapsto \begin{pmatrix} a & 0 & b & ay-bx \\ x & 1 & y & xy \\ c & 0 & d & cy-dx \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix on the right-hand side is the matrix product of the following embeddings of Γ and $\mathbb{Z}^2$ into $\mathrm{Sp}_2(\mathbb{Z})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (x,y) \mapsto \begin{pmatrix} 1 & 0 & 0 & y \\ x & 1 & y & xy \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For every pair (τ, z) in $\mathfrak{H} \times \mathbb{C}$, let $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)$ be an element of $\mathfrak{H}_2$ (it follows that ω is a variable in $\mathfrak{H}$). It is easy to check that $\phi(\tau, z) \in J_{k,m}$ if and only if $\psi(Z) = \phi(\tau, z)e(m\omega)$ is a Siegel modular form of weight k and degree 2 for the parabolic subgroup (1.28). For every positive-definite, even lattice $\underline{L}$, there exist canonical injective homomorphisms $H^{\underline{L}}(\mathbb{Z}) \hookrightarrow J^{\underline{L}}$ and $\Gamma \hookrightarrow J^{\underline{L}}$, given by

$$h \mapsto (I_2, h) \qquad \text{and} \qquad A \mapsto (A, (0,0)).$$

In particular, it follows that

$$\Gamma \hookrightarrow \Gamma^J \hookrightarrow \mathrm{Sp}_2(\mathbb{Z}).$$

It is straight-forward to check that $J_{k,0} = M_k(\Gamma)$. It follows that Jacobi forms of scalar index are an intermediate between elliptic modular forms and Siegel modular forms of degree two.

It was proved in [**38**] that there exists a *Hecke equivariant lifting map* between Jacobi forms and elliptic modular forms. Let W_m denote the m-th Atkin–Lehner involution $\left(\begin{smallmatrix} 0 & -1 \\ m & 0 \end{smallmatrix}\right)$ and set

$$M_k^\varepsilon(m) := \mathrm{Span}_{\mathbb{C}}\{f \in M_k(m) : f|_k W_m = \varepsilon i^{-k} f\},$$

where $\varepsilon \in \{+,-\}$. Then

$$f \in M_k^\varepsilon(m) \implies \Lambda_m(s,f) = \varepsilon \Lambda_m(k-s,f).$$

The space $M_k(m)$ has a (not necessarily unique) basis of modular forms whose L-series have an Euler product. Every such modular form f is an eigenform of all Hecke operators $T(l)$ with $(l,m) = 1$ and has the same eigenvalues for these operators as a unique newform g in $M_k(m')$ for some $m' \mid m$. The quotient $L(f,s)/L(g,s)$ is a finite Dirichlet series with an product expansion of the form

$$\frac{L(f,s)}{L(g,s)} = \prod_{p \mid \frac{m}{m'}} Q_p(s),$$

where $Q_p(s)$ is a polynomial in p^{-s}. The L-series of g has a functional equation under $s \mapsto k-s$ and, provided f is an eigenform of all Atkin–Lehner involutions on $M_k(m)$, so does $L(f,s)$. It follows that each of the Q_p's has a functional equation

$$Q_p(k-s) = \pm p^{-v_p(m'/m)(k-2s)} Q_p(s).$$

If $f \in M_k^-(m)$ and even one of the signs in the above functional equation is a minus, then $L(f,s)$ has a higher order of vanishing at $s = \frac{k}{2}$ than $L(g,s)$. Define $\mathfrak{M}_k(m)$ to be the subspace of $M_k(m)$ which is spanned by all f for which the sign in the above equation is + for all $p \mid \frac{m}{m'}$ and set

$$\mathfrak{M}_k^{\pm}(m) := \mathfrak{M}_k(m) \cap M_k^{\pm}(m).$$

The following holds:

THEOREM 1.43. *For $k \geq 2$, the spaces $J_{k,m}$ and $\mathfrak{M}_{2k-2}^-(m)$ are isomorphic as Hecke modules.*

The lifting map is given below:

THEOREM 1.44. *Let D be a fundamental discriminant and let s be an integer such that $D \equiv s^2 \bmod 4m$. Then the map*

$$\mathscr{S}_{D,s} : J_{k,m} \to \mathfrak{M}_{2k-2}^-(m),$$

defined by

$$\sum_{\substack{n,r' \in \mathbb{Z} \\ 4mn - r'^2 \geq 0}} b(n,r') q^n \zeta^{r'} \mapsto \sum_{l \geq 0} \Big\{ \sum_{a \mid l} a^{k-2} \left(\tfrac{D}{a}\right) b\left(\frac{l^2}{a^2} \cdot \frac{D-s^2}{4m}, \frac{l}{a} s\right) \Big\} q^l$$

commutes with Hecke operators and with Atkin–Lehner involutions, it preserves cusp forms and Eisenstein series and a linear combination of these maps is an isomorphism.

Special care needs to be taken when $l = 0$ in the above equation, however we omit the details and refer the reader to [**38**, §3] instead.

1.2.4.2. *Jacobi forms and Siegel modular forms.* We generalize the construction from Remark 1.42 for every arbitrary positive-definite, even lattice $\underline{L} = (L, \beta)$.

Fix a $\mathbb{Z}$-basis of L. It follows that the Gram matrix G of $\underline{L}$ is also fixed and the vector coordinates of elements of L depend on this basis as well. Embed Γ into $\mathrm{Sp}_{\mathrm{rk}(\underline{L})+1}(\mathbb{Z})$ in the following way:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \check{A} := \begin{pmatrix} a & \mathbf{0}^t & b & \mathbf{0}^t \\ 0 & I_{\mathrm{rk}(\underline{L})} & \mathbf{0} & \mathbf{0}_{\mathrm{rk}(\underline{L})} \\ c & \mathbf{0}^t & d & \mathbf{0}^t \\ \mathbf{0} & \mathbf{0}_{\mathrm{rk}(\underline{L})} & \mathbf{0} & I_{\mathrm{rk}(\underline{L})} \end{pmatrix},$$

where $\mathbf{0}$ is the $\mathrm{rk}(\underline{L}) \times 1$ zero vector and $\mathbf{0}_{\mathrm{rk}(\underline{L})}$ is the $\mathrm{rk}(\underline{L}) \times \mathrm{rk}(\underline{L})$ zero matrix. Embed $H^{\underline{L}}(\mathbb{Z})$ into $\mathrm{Sp}_{\mathrm{rk}(\underline{L})+1}(\mathbb{Z})$ in the following way:

$$h = (\lambda, \mu) \mapsto \check{h} := \begin{pmatrix} 1 & \mathbf{0}^t & 0 & \mu^t \\ \lambda & I_{\mathrm{rk}(\underline{L})} & \mu & \lambda\mu^t \\ 0 & \mathbf{0}^t & 1 & -\lambda^t \\ \mathbf{0} & \mathbf{0}_{\mathrm{rk}(\underline{L})} & \mathbf{0} & I_{\mathrm{rk}(\underline{L})} \end{pmatrix}.$$

It is straight-forward to check that $\check{A}$ and $\check{h}$ are elements of $\mathrm{Sp}_{\mathrm{rk}(\underline{L})+1}(\mathbb{Z})$. Finally, embed $J^{\underline{L}}$ into $\mathrm{Sp}_{\mathrm{rk}(\underline{L})+1}(\mathbb{Z})$ by taking the matrix product of the above embeddings:

$$\gamma = (A, h) \mapsto \check{\gamma} := \check{A}\check{h} = \begin{pmatrix} a & \mathbf{0}^t & b & a\mu^t - b\lambda^t \\ \lambda & I_{\mathrm{rk}(\underline{L})} & \mu & \lambda\mu^t \\ c & \mathbf{0}^t & d & c\mu^t - d\lambda^t \\ \mathbf{0} & \mathbf{0}_{\mathrm{rk}(\underline{L})} & \mathbf{0} & I_{\mathrm{rk}(\underline{L})} \end{pmatrix}. \tag{1.29}$$

Let $\Gamma^J_{\mathrm{rk}(\underline{L})}$ denote the image of this embedding.

Next, embed $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ into $\mathfrak{H}_{\mathrm{rk}(\underline{L})+1}$: identify every element z in $L \otimes_{\mathbb{Z}} \mathbb{C}$ with its coefficient vector with respect to our chosen basis and complete every pair (τ, z) in $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ with a variable ω in $\mathfrak{H}_{\mathrm{rk}(\underline{L})}$ such that $\mathfrak{I}(\tau)\mathfrak{I}(\omega) - \mathfrak{I}(z)\mathfrak{I}(z^t) > 0$ in the following way:

$$(\tau, z) \mapsto Z := \begin{pmatrix} \tau & z^t \\ z & \omega \end{pmatrix}.$$

It is well-known that $Z \in \mathfrak{H}_{\mathrm{rk}(\underline{L})+1}$ [**2**]. For every holomorphic function $\phi : \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$, define a lift

$$\check{\phi}(Z) := \phi(\tau, z)\exp(\pi i \,\mathrm{Tr}(\omega G)).$$

This is not a Siegel modular form,

The following holds:

Lemma 1.45. *A holomorphic function $\phi : \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ is an element of $J_{k,\underline{L}}$ if and only if its lift $\check{\phi} : \mathfrak{H}_{\mathrm{rk}(\underline{L})+1} \to \mathbb{C}$ is a Siegel modular form of weight k and degree $\mathrm{rk}(\underline{L}) + 1$ for $\Gamma^J_{\mathrm{rk}(\underline{L})}$.*

I think this is simply wrong!

Proof. The holomorphic function $\check{\phi}$ is a Siegel modular form of weight k and degree $\mathrm{rk}(\underline{L}) + 1$ for $\Gamma^J_{\mathrm{rk}(\underline{L})}$ if and only if

$$\check{\phi}(\check{\gamma}Z) = \det(PZ + Q)^k \check{\phi}(Z) \tag{1.30}$$

for every $\check{\gamma} = \left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix}\right)$ as in (1.29). We have

Again. Since this subsection contains various wrong statements just suppress it. As it stands it is not helpful but only confusing.

$$MZ + N = \begin{pmatrix} a\tau + b & a(z+\mu)^t - b\lambda^t \\ z + \lambda\tau + \mu & \lambda z^t + \omega + \lambda\mu^t \end{pmatrix}$$

and

$$PZ + Q = \begin{pmatrix} c\tau + d & c(z+\mu)^t - d\lambda^t \\ \mathbf{0} & I_{\mathrm{rk}(\underline{L})} \end{pmatrix}.$$

It follows that

$$(PZ + Q)^{-1} = \begin{pmatrix} (c\tau + d)^{-1} & \frac{-c(z+\mu)^t + d\lambda^t}{c\tau + d} \\ \mathbf{0} & I_{\mathrm{rk}(\underline{L})} \end{pmatrix},$$

which implies that $\det(PZ + Q)^{-k} = (c\tau + d)^{-k}$ and hence that

$$\check{\gamma}Z = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & \frac{a\tau+b}{c\tau+d}(-c(z+\mu) + d\lambda)^t + a(z+\mu)^t - b\lambda^t \\ \frac{z+\lambda\tau+\mu}{c\tau+d} & \frac{(z+\lambda\tau+\mu)(-c(z+\mu)+d\lambda)^t}{c\tau+d} + \lambda z^t + \omega + \lambda\mu^t \end{pmatrix}.$$

Note that

$$\frac{a\tau + b}{c\tau + d}(-c(z+\mu) + d\lambda) + a(z+\mu) - b\lambda = \frac{z + \lambda\tau + \mu}{c\tau + d}$$

and

$$\frac{(z + \lambda\tau + \mu)(-c(z+\mu) + d\lambda)^t}{c\tau + d} = -\frac{c(z + \lambda\tau + \mu)(z + \lambda\tau + \mu)^t}{c\tau + d} + z\lambda^t + \tau\lambda\lambda^t + \mu\lambda^t$$

and therefore

$$\check{\gamma}Z = \begin{pmatrix} A\tau & \frac{(z+\lambda\tau+\mu)^t}{c\tau+d} \\ \frac{z+\lambda\tau+\mu}{c\tau+d} & -\frac{c(z+\lambda\tau+\mu)(z+\lambda\tau+\mu)^t}{c\tau+d} + (z\lambda^t+\lambda z^t)+\tau\lambda\lambda^t+(\mu\lambda^t+\lambda\mu^t)+\omega \end{pmatrix}.$$

Let $\omega(Z)$ denote the bottom right entry of every element Z of $\mathfrak{H}_{\mathrm{rk}(\underline{L})+1}$. It follows that (1.30) is equivalent to

$$\phi(\tau, z)\exp\left(\pi i \operatorname{Tr}(\omega(Z)G)\right) = (c\tau + d)^{-k}\phi\Big(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\Big)\exp\left(\pi i \operatorname{Tr}(\omega(\check{\gamma}Z)G)\right)$$

and this is in turn equivalent to

$$\begin{aligned}\phi(\tau, z) =&(c\tau + d)^{-k}\phi\Big(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\Big)\exp\Big(\pi i \operatorname{Tr}\Big(\Big((z\lambda^t + \lambda z^t) + \tau\lambda\lambda^t \\ &+ (\mu\lambda^t + \lambda\mu^t) - \frac{c(z+\lambda\tau+\mu)(z+\lambda\tau+\mu)^t}{c\tau+d}\Big)G\Big)\Big).\end{aligned} \tag{1.31}$$

For every $\mathrm{rk}(\underline{L}) \times 1$ vector v, we have

$$\begin{aligned}\operatorname{Tr}((vv^t)G) &= \operatorname{Tr}\begin{pmatrix} v_1\sum_i v_i g_{i1} & v_1\sum_i v_i g_{i2} & \dots & v_1\sum_i v_i g_{i\mathrm{rk}(\underline{L})} \\ v_2\sum_i v_i g_{i1} & v_2\sum_i v_i g_{i2} & \dots & v_2\sum_i v_i g_{i\mathrm{rk}(\underline{L})} \\ \dots & \dots & \dots & \dots \\ v_{\mathrm{rk}(\underline{L})}\sum_i v_i g_{i1} & v_{\mathrm{rk}(\underline{L})}\sum_i v_i g_{i2} & \dots & v_{\mathrm{rk}(\underline{L})}\sum_i v_i g_{i\mathrm{rk}(\underline{L})} \end{pmatrix} \\ &= \sum_{i,j} v_i v_j g_{ij} = 2\beta(v).\end{aligned}$$

Since $\operatorname{Tr}(MG) = \operatorname{Tr}(M^t G)$ for every $\mathrm{rk}(\underline{L}) \times \mathrm{rk}(\underline{L})$ matrix M, we have

$$\begin{aligned}\operatorname{Tr}((vw^t + wv^t)G) =&2\operatorname{Tr}\begin{pmatrix} v_1\sum_i w_i g_{i1} & \dots & v_1\sum_i w_i g_{i\mathrm{rk}(\underline{L})} \\ v_2\sum_i w_i g_{i1} & \dots & v_2\sum_i w_i g_{i\mathrm{rk}(\underline{L})} \\ \dots & \dots & \dots \\ v_{\mathrm{rk}(\underline{L})}\sum_i w_i g_{i1} & \dots & v_{\mathrm{rk}(\underline{L})}\sum_i w_i g_{i\mathrm{rk}(\underline{L})} \end{pmatrix} \\ =&2\sum_{i,j} w_i v_j g_{ij} = 2\beta(v, w)\end{aligned}$$

for every two $\mathrm{rk}(\underline{L}) \times 1$ vectors v and w. Thus, the exponential factor on the right-hand side of (1.31) is equal to

$$e\Big(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda, z)\Big),$$

which is the exponential factor from the definition of the $|_{k,\underline{L}}$-action of the Jacobi group (1.10).

Last, but not least, the Siegel–Fourier expansion of $\check{\phi}$ is equivalent to the Fourier expansion of ϕ. The group $\Gamma^J_{\mathrm{rk}(\underline{L})}$ contains matrices of the form

$$\begin{pmatrix} I_{\mathrm{rk}(\underline{L})+1} & T \\ \mathbf{0}_{\mathrm{rk}(\underline{L})} & I_{\mathrm{rk}(\underline{L})+1} \end{pmatrix},$$

where T is a positive semi-definite, even, $(\mathrm{rk}(\underline{L})+1)\times(\mathrm{rk}(\underline{L})+1)$ matrix. It follows that every Siegel modular form of degree $\mathrm{rk}(\underline{L}) + 1$ and weight k for $\Gamma^J_{\mathrm{rk}(\underline{L})}$ is invariant with respect to the linear transformations $Z \mapsto Z + T$ given by such matrices and hence has a Fourier expansion of the form:

$$\check{\phi}(Z) = \sum_{T\geq 0} A(T)\exp(\pi i \operatorname{Tr}(ZT)),$$

where the summation is taken over all positive semi-definite, even, $(\mathrm{rk}(\underline{L})+1)\times(\mathrm{rk}(\underline{L})+1)$ matrices. The Fourier coefficients $A(T)$ are given by the integral

$$A(T) = \int_{X \bmod 1} \psi(Z)\exp(-\pi i\,\mathrm{Tr}(ZT))dX,$$

where we have written $Z = X + iY$ and dX is the Euclidean volume of the space of X-coordinates. Write every Z in $\mathfrak{H}_{\mathrm{rk}(\underline{L})+1}$ as $Z = \left(\begin{smallmatrix}\tau & z^t\\ z & \omega\end{smallmatrix}\right)$, with τ in $\mathfrak{H}$, ω in $\mathfrak{H}_{\mathrm{rk}(\underline{L})}$ and $z \in \mathbb{C}^{\mathrm{rk}(\underline{L})}$, such that $\Im(\tau)\Im(\omega) - \Im(z)\Im(z^t) > 0$. Every T as above can be written as

$$T = \begin{pmatrix}2n & x^t\\ x & M\end{pmatrix} = \begin{pmatrix}1 & \mathbf{0}^t\\ M^{-1}x & I_{\mathrm{rk}(\underline{L})}\end{pmatrix}^t \begin{pmatrix}2n - x^tM^{-1}x & \mathbf{0}^t\\ \mathbf{0} & M\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}^t\\ M^{-1}x & I_{\mathrm{rk}(\underline{L})}\end{pmatrix},$$

with n in $\mathbb{Z}$, M a positive semi-definite, even, $\mathrm{rk}(\underline{L}) \times \mathrm{rk}(\underline{L})$ matrix and x in $\mathbb{Z}^{\mathrm{rk}(\underline{L})}$ such that $2n - x^tM^{-1}x \geq 0$. This is called the *Jacobi decomposition* of T. Writing $A(T) = A_M(n, x)$, the Fourier expansion of $\check{\phi}$ becomes

$$\begin{aligned}\check{\phi}(Z) &= \sum_{\substack{n,x,M\\ 2n-x^tM^{-1}x\geq 0}} A_M(n,x)\exp(\pi i\,(2n\tau + \mathrm{Tr}(2zx^t + \omega M)))\\ &= \sum_{M\geq 0}\Bigg(\sum_{\substack{n,x\\ 2n-x^tM^{-1}x\geq 0}} A_M(n,x)e\left(n\tau + \mathrm{Tr}(zx^t)\right)\Bigg)\exp(\pi i\,\mathrm{Tr}(\omega M)).\end{aligned}$$

Since $\check{\phi}(Z) = \phi(\tau, z)\exp(\pi i\,\mathrm{Tr}(\omega G))$, it follows that

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, x\in\mathbb{C}^{\mathrm{rk}(\underline{L})}\\ 2n-x^tG^{-1}x\geq 0}} A_G(n,x)e\left(n\tau + \mathrm{Tr}(zx^t)\right).$$

Substitute $Gr = x$. Then $x \in \mathbb{Z}^{\mathrm{rk}(\underline{L})} \iff r \in L^{\#}$ and $\mathrm{Tr}(zx^t) = \beta(r, z)$. Furthermore, $2n - x^tG^{-1}x \geq 0 \iff n \geq \beta(r)$. Hence,

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^{\#}\\ n\geq\beta(y)}} A_G(n, G\tilde{r})e\left(n\tau + \beta(r,z)\right),$$

completing the proof. □

1.2.4.3. *Jacobi forms and vector-valued modular forms.* In this subsection, we discuss the connection between Jacobi forms and vector-valued modular forms for the dual Weil representation (see Definition 1.4 and Definition 1.16). We remind the reader of the theta expansion

$$\phi(\tau, z) = \sum_{x\in L^{\#}/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau, z)$$

of a Jacobi form ϕ in $J_{k,\underline{L}}$. It was proved in [**3**, §3.7] that, for every $x \in L^{\#}/L$ and every $\tilde{A}$ in $\tilde{\Gamma}$, the functions $h_{\phi,x}$ satisfy the following:

$$h_{\phi,x}|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A} = \sum_{y\in L^{\#}/L}\overline{\rho_{\underline{L}}(\tilde{A})_{x,y}}h_{\phi,y}.$$

These modular properties imply that the vector-valued function

$$h_{\phi,\underline{L}}(\tau) := \sum_{x\in L^{\#}/L} h_{\phi,x}(\tau)\mathfrak{e}_x$$

This was defined before. Maybe one should add in parenthesis: "(see Section ... for the definition of the latter)".

is an element of $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}}^*)$. Moreover, as a result of (1.19), the vector-valued function

$$\theta_{\underline{L}}(\tau, z) := \sum_{x \in L^\#/L} \vartheta_{\underline{L},x}(\tau, z)$$

is an element of $M_{\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}})$. The main result in [**3**, §3] is the following theorem:

Theorem 1.46. *If $\underline{L} = (L, \beta)$ is a positive-definite, even lattice over $\mathbb{Z}$, then the map*

$$\varphi : \phi \mapsto h_{\phi,\underline{L}}$$

is an isomorphism between $J_{k,\underline{L}}$ and $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}}^*)$.

The results in [**3**] hold over arbitrary totally real number fields, not only over $\mathbb{Q}$. A consequence of this theorem is that $J_{k,\underline{L}} = \{0\}$ if $k < \mathrm{rk}(\underline{L})/2$ and that the spaces $J_{k,\underline{L}}$ are finite-dimensional. When $k \in \mathbb{Z}$, it also gives a connection between Jacobi forms of odd rank lattice index and half-integral weight elliptic modular forms, while for Jacobi forms of even rank lattice index it gives a connection to integral weight elliptic modular forms. For every fixed lattice $\underline{L}$, the value $k = \mathrm{rk}(\underline{L})/2$ is called its *singular weight*. The value $k = (\mathrm{rk}(\underline{L}) + 1)/2$ is called the *critical weight*. Note that there also exists an isomorphism between *skew-holomorphic* Jacobi forms of lattice index $\underline{L}$ and vector-valued modular forms for $\rho_{\underline{L}}$. We do not go into further details and instead refer the reader to [**10**, §15.2], for example, where the scalar case is treated.

Another important representation in the theory of vector-valued modular forms is the *Schrödinger representation*. It is typically a representation of the Heisenberg group on the group algebra $\mathbb{C}[D]$ of some finite quadratic module D. Let H be the Heisenberg group $\mathbb{Z}^3$ with the following composition law:

$$(m, n, t)(m', n', t') = (m + m', n + n', t + t' + mn' - nm').$$

Definition 1.47 (Schrödinger representation). Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $x \in L^\#/L$. The Schrödinger representation of H on $\mathbb{C}[L^\#/L]$ twisted at x is the representation $\sigma_x : H \to \mathrm{Aut}(\mathbb{C}[L^\#/L])$ defined by

$$\sigma_x(m, n, t)\mathfrak{e}_y := e\left(n\beta(x, y) + (t - mn)\beta(x)\right)\mathfrak{e}_{y-mx}.$$

We check that σ_x is indeed a representation: we have $\sigma_x(0, 0, 0) = I_{\det(\underline{L})}$ and, for arbitrary elements (m, n, t) and (m', n', t') of H, we have

$$\begin{aligned}(\sigma_x(m, n, t)\sigma_x(m', n', t'))\,\mathfrak{e}_y =& \sigma_x(m, n, t)e\left(n'\beta(x, y) + (t' - m'n')\beta(x)\right)\mathfrak{e}_{y-m'x}\\ =& e\left(n'\beta(x, y) + (t' - m'n')\beta(x)\right)\\ & \times e\left(n\beta(x, y - m'x) + (t - mn)\beta(x)\right)\mathfrak{e}_{y-m'x-mx}\\ =& e((n + n')\beta(x, y) + (t + t' + mn' - nm'\\ & - (m + m')(n + n')\beta(x))\mathfrak{e}_{y-(m+m')x}\\ =& \sigma_x\left((m, n, t)(m', n', t')\right)\mathfrak{e}_y.\end{aligned}$$

Every element (m, n, t) of H can be written as a product

$$(m, 0, 0)(0, n, 0)(0, 0, t).$$

We remind the reader that a representation $\pi : G \to \mathrm{Aut}(V)$ is unitary if and only if $\overline{\pi(g)}^t\pi(g) = I_{\dim(V)}$ for all g in G. Let $\{y_1, \ldots, y_{\det(\underline{L})}\}$ denote the elements of $L^\#/L$. Then $\sigma_x(1, 0, 0)\mathfrak{e}_{y_i} = \mathfrak{e}_{y_i - x}$ and therefore the matrix of $\sigma_x(1, 0, 0)$ is a permutation matrix (hence it is unitary). Furthermore, $\sigma_x(0, 1, 0)\mathfrak{e}_{y_i} = e(\beta(x, y_i))\mathfrak{e}_{y_i}$ and $\sigma_x(0, 0, 1)\mathfrak{e}_{y_i} = e(\beta(x))\mathfrak{e}_{y_i}$, therefore their matrices are diagonal with diagonal entries of modulus equal to one (hence they are unitary). It follows that σ_x is unitary.

Define the following right-action of Γ on H, for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ:

$$((m,n,t),A) \mapsto (m,n,t)^A := (ma+nc, mb+nd, t).$$

Lemma 1.48. *For every $\tilde{A}$ in $\tilde{\Gamma}$ and every (m,n,t) in H, the following relation holds between the Weil and the Schrödinger representations:*

$$\rho_{\underline{L}}(\tilde{A})^{-1}\sigma_x(m,n,t)\rho_{\underline{L}}(\tilde{A}) = \sigma_x((m,n,t)^A). \tag{1.32}$$

Proof. Check that (1.32) holds for the generators $\tilde{T}$ and $\tilde{S}$ of $\tilde{\Gamma}$:

$$\begin{aligned}
\rho_{\underline{L}}(\tilde{T})^{-1}\sigma_x(m,n,t)\rho_{\underline{L}}(\tilde{T})\mathfrak{e}_y =& \rho_{\underline{L}}(\tilde{T})^{-1}\sigma_x(m,n,t)e(\beta(y))\mathfrak{e}_y \\
=& \rho_{\underline{L}}(\tilde{T})^{-1}e\left(n\beta(x,y)+(t-mn)\beta(x)+\beta(y)\right)\mathfrak{e}_{y-mx} \\
=& e\left(n\beta(x,y)+(t-mn)\beta(x)+\beta(y)-\beta(y-mx)\right)\mathfrak{e}_{y-mx} \\
=& e((m+n)\beta(x,y)+(t-m(m+n))\beta(x))\mathfrak{e}_{y-mx} \\
=& \sigma_x((m,n,t)^T)\mathfrak{e}_y.
\end{aligned}$$

For $\tilde{S}$, it suffices to check that equality holds for the three generators of H. We include the calculations in one of the three cases, since the rest can be treated analogously:

$$\begin{aligned}
\rho_{\underline{L}}(\tilde{S})^{-1}\sigma_x(1,0,0)\rho_{\underline{L}}(\tilde{S})\mathfrak{e}_y =& \rho_{\underline{L}}(\tilde{S})^{-1}\sigma_x(1,0,0)\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}}\sum_{s\in L^\#/L} e(-\beta(y,s))\mathfrak{e}_s \\
=& \rho_{\underline{L}}(\tilde{S})^{-1}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}}\sum_{s\in L^\#/L} e(-\beta(y,s))\mathfrak{e}_{s-x} \\
=& \frac{1}{\det(\underline{L})}\sum_{s\in L^\#/L}\sum_{r\in L^\#/L} e(\beta(s-x,r)-\beta(y,s))\mathfrak{e}_r \\
=& \frac{1}{\det(\underline{L})}\sum_{r\in L^\#/L} e(-\beta(x,r))\mathfrak{e}_r\sum_{s\in L^\#/L} e(\beta(r-y,s)) \\
=& e(-\beta(x,y))\mathfrak{e}_y = \sigma_x((1,0,0)^S)\mathfrak{e}_y,
\end{aligned}$$

where we have used the fact that, for every y in $L^\#/L$, we have

$$\sum_{s\in L^\#/L} e(\beta(y,s)) = \begin{cases} \det(\underline{L}), & \text{if } y=0 \text{ and} \\ 0, & \text{otherwise.}\end{cases}$$

□

Since σ_x is unitary, its dual representation is obtained by complex conjugation:

$$\sigma_x^*(m,n,t)\mathfrak{e}_y = e\left(-n\beta(x,y)+(mn-t)\beta(x)\right)\mathfrak{e}_{y-mx}.$$

Taking complex conjugates on both sides of (1.32), we obtain the following relation between the duals of the Schrödinger and the Weil representations:

$$\overline{\rho_{\underline{L}}(\tilde{A})^{-1}}\sigma_x^*(m,n,t)\rho_{\underline{L}}^*(\tilde{A}) = \sigma_x^*((m,n,t)^A)$$

and therefore

$$\sigma_x^*(m,n,t) = \rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,t)^A)\rho_{\underline{L}}^*(\tilde{A})^{-1}. \tag{1.33}$$

We will use the Schrödinger representation in Section 2.4 to define an *averaging operator* on $J_{k,\underline{L}}$.

1.2.4.4. *Jacobi forms and orthogonal modular forms.* The connection between Jacobi forms and orthogonal modular forms bears similarities to that between the former and Siegel modular forms. It is, however, more natural, in the sense that it does not depend on the choice of a basis. Orthogonal modular forms have many applications in algebraic geometry. For example, they can be the automorphic discriminants of moduli spaces [**17**], which allows for the construction of modular varieties [**21**].

Let $(L_0, (\cdot,\cdot)_0)$ be an even lattice of signature $(2, g+2)$, which contains two hyperbolic planes:

$$L_0 = U_0 \oplus U_1 \oplus L(-1),$$

where L is a positive-definite, even lattice of rank g with bilinear form $(\cdot,\cdot)$. Set $r^2 = \frac{1}{2}(r,r)_0$ for every r in $L \otimes_{\mathbb{Z}} \mathbb{C}$. Fix a $\mathbb{Z}$-basis $\{e_0, e_1, \ldots, e_{g+2}, e_{g+3}\}$ of L_0 such that $U_0 = \mathbb{Z}e_0 \oplus \mathbb{Z}e_{g+3}$, $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_{g+2}$ and $\{\ldots\}$ is a basis of L. In other words, we have $e_0^2 = e_1^2 = e_{g+2}^2 = e_{g+3}^2 = 0$ and $(e_0, e_{g+3})_0 = (e_1, e_{g+2})_0 = 1$. The Gram matrix of $(\cdot,\cdot)_0$ with respect to this basis is then equal to

$$\left(\begin{array}{cc|ccc|cc}
0 & 0 & 0 & \ldots & 0 & 0 & 1\\
0 & 0 & 0 & \ldots & 0 & 1 & 0\\
\hline
0 & 0 & & & & 0 & 0\\
\vdots & \vdots & & -G & & \vdots & \vdots\\
0 & 0 & & & & 0 & 0\\
\hline
0 & 1 & 0 & \ldots & 0 & 0 & 0\\
1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right),$$

where G denotes the Gram matrix of L. Let $\mathcal{D}(L_0)$ denote the $(g+2)$-dimensional bounded symmetric Hermitian domain of type IV associated with L_0, i.e. one of the two connected components of the set

$$\left\{[\mathcal{Z}] \in \mathbb{P}(L_0 \otimes_{\mathbb{Z}} \mathbb{C}) : (\mathcal{Z}, \mathcal{Z})_0 = 0, (\mathcal{Z}, \overline{\mathcal{Z}})_0 > 0\right\}.$$

The two connected components of this set are mapped isomorphically to each other by complex conjugation. Let $O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})$ denote the index 2 subgroup of the real orthogonal group which preserves $\mathcal{D}(L_0)$ (it is the connected component of the identity of the real orthogonal group of L_0). The "+" in the notation marks our chosen connected component. Denote by $O^+(L_0)$ the stabilizer of L_0 inside the subgroup $O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})$, i.e. the intersection $O(L_0) \cap O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})$. This is an arithmetic group.

The domain $\mathcal{D}(L_0)$ can be realized as a tube domain inside $\mathbb{C}^{g+2}$. Let F denote the totally isotropic plane spanned by e_0 and e_1, set $L_1 := \mathbb{Z}e_1 \oplus L \oplus \mathbb{Z}e_{g+2}$ and define

$$\mathcal{H}(L_0) := \left\{ Z = \begin{pmatrix} \omega \\ z \\ \tau \end{pmatrix} \in \mathfrak{H} \times L \otimes_{\mathbb{Z}} \mathbb{C} \times \mathfrak{H} : \frac{1}{2}\left(\Im(Z), \Im(Z)\right)_1 > 0 \right\},$$

where $(\cdot,\cdot)_1$ denotes the restriction of $(\cdot,\cdot)_0$ to L_1. Note that $(Z,Z)_1 = 2\omega\tau - (z,z)$. The embedding

$$\mathrm{pr} : Z \mapsto \mathrm{pr}(Z) := \begin{pmatrix} -\frac{1}{2}(Z,Z)_1 \\ \omega \\ z \\ \tau \\ 1 \end{pmatrix}$$

of $\mathcal{H}(L_0)$ into $\mathbb{P}(L_0 \otimes \mathbb{C})$ defines an isomorphism between $\mathcal{H}(L_0)$ and $\mathcal{D}(L_0)$ depending on F (and note that F corresponds to a one-dimensional cusp in the modular variety $SO^+(L_0)\backslash\mathcal{D}(L_0)$).

The group $O^+(L_0)$ acts on $\mathcal{H}(L_0)$ from the left via

$$(\gamma, Z) \mapsto \gamma\langle Z\rangle := \mathrm{pr}^{-1}\left(\frac{\gamma \mathrm{pr}(Z)}{J(\gamma, Z)}\right),$$

where the holomorphic automorphy factor on $\mathcal{H}(L_0)$ with respect to $O^+(L_0)$ is defined as the bottom entry of $\gamma\mathrm{pr}(Z)$:

$$J(\gamma, Z) := -\frac{1}{2}\gamma_{g+4,1}(Z,Z)_1 + \gamma_{g+4,2}\omega + \sum_{j=3}^{g+2}\gamma_{g+4,j}z_{j-2} + \gamma_{g+4,g+3}\tau + \gamma_{g+4,g+4}.$$

For every integer k, define a right-action of $O^+(L_0)$ on the space of functions $\psi : \mathcal{H}(L_0) \to \mathbb{C}$ in the following way:

$$(\psi, \gamma) \mapsto \psi|_k\gamma(Z) := J(\gamma, Z)^{-k}\psi(\gamma\langle Z\rangle).$$

A *modular form of weight k with respect to* $O^+(L_0)$ is a holomorphic function $\psi : \mathcal{H}(L_0) \to \mathbb{C}$ which satisfies

$$\psi|_k\gamma(Z) = \psi(Z) \tag{1.34}$$

for every γ in $O^+(L_0)$. An analogous definition works to define orthogonal modular forms with respect to every finite index subgroup of $O^+(L_0)$.

Since the orthogonal group $O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})$ has rank equal to 2, there are two types of maximal parabolic subgroups in $O^+(L_0)$ and hence two types of Fourier expansions of orthogonal modular forms:

$$\psi(Z) = \sum_{\substack{\lambda \in L_1^\# \\ i\lambda \in \mathcal{H}(L_0), (\lambda,\lambda)_1 \geq 0}} f(\lambda) e((\lambda, Z)_1) \text{ and}$$

$$\psi\begin{pmatrix}\omega \\ z \\ \tau\end{pmatrix} = \sum_{m \geq 0}\phi_m(\tau, z)e(m\omega).$$

The latter is called the *Fourier-Jacobi expansion* of an orthogonal modular form.

Let P_F denote the parabolic subgroup of $O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})$ which preserves F, i.e.

$$P_F := \left\{\begin{pmatrix} A^* & B_1 & E_2T \\ 0 & U & B \\ 0 & 0 & A\end{pmatrix} \in O^+(L_0 \otimes_{\mathbb{Z}} \mathbb{R})\right\},$$

$$A \in \mathrm{GL}_2^+(\mathbb{R}), U \in O^+(L \otimes \mathbb{R}), B \in \mathrm{M}_{g,2}(\mathbb{R}), A^* = E_2(A^t)^{-1}$$
$$E_2, B_1 = E_2(A^t)^{-1}B^tGU, T^tA + A^tT = B^tGB.$$

The *real Jacobi group* $O^J_{\mathbb{R}}$ is the subgroup of P_F which acts trivially on L. It has no anisotropic part U and it is generated by the following elements:

$$\{A\} := \mathrm{diag}(A^*, I_g, A), A \in \mathrm{SL}_2(\mathbb{R}) \text{ and}$$

$$\{x, y; r\} := \begin{pmatrix} 1 & 0 & y^tG & (x,y) - r & \frac{1}{2}(y,y) \\ 0 & 1 & x^tG & \frac{1}{2}(x,x) & r \\ 0 & 0 & I_g & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

where $x, y \in M_{g,1}(\mathbb{R})$ and $r \in \mathbb{R}$. It is clear that $\{A\} \in P_F$. To check that $\{x, y; r\} \in P_F$ as well, consider $B = (x, y)$ and $A = E_2$ in the above, in order to obtain that $B_1 = E_2 B^t G = \left(\begin{smallmatrix} y^t G \\ x^t G \end{smallmatrix}\right)$ and that

$$T' + T = B^t G B = \begin{pmatrix} x^t G x & x^t G y \\ y^t G x & y^t G y \end{pmatrix}.$$

Write $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and therefore $a = \frac{1}{2}(x, x)$, $d = \frac{1}{2}(y, y)$, $b = r$ and $c = (x, y) - r$ and hence $\{x, y; r\} \in P_F$. The Jacobi group has a subgroup isomorphic to Γ (the isomorphism is given by the map $A \mapsto \{A\}$). By abuse of notation, we denote this subgroup by Γ. The elements $\{x, y; r\}$ act trivially on F. They generate a normal subgroup of $O^J_{\mathbb{R}}$ which is isomorphic to the Heisenberg group $H(L \otimes_{\mathbb{Z}} \mathbb{R})$ of dimension $2g + 1$. The center of $H(L \otimes_{\mathbb{Z}} \mathbb{R})$ is equal to $\{\{0, 0; t\} : t \in \mathbb{R}\}$. Hence, the Heisenberg group is the central extension of $(L \otimes_{\mathbb{Z}} \mathbb{R})^2$:

$$0 \to \mathbb{R} \to H(L \otimes_{\mathbb{Z}} \mathbb{R}) \to (L \otimes_{\mathbb{Z}} \mathbb{R}) \times (L \otimes_{\mathbb{Z}} \mathbb{R}) \to 0.$$

The modular group $SL_2(\mathbb{R})$ acts on $H(L \otimes_{\mathbb{Z}} \mathbb{R})$ by conjugation:

$$\{x, y; r\} \mapsto A \cdot \{x, y; r\} := \{A\}\{x, y; r\}\{A^{-1}\}.$$

The *integral Jacobi group* O^J is the intersection $O^J_{\mathbb{R}} \cap O^+(L_0)$ and it is isomorphic to the semidirect product $\Gamma \ltimes H(L)$. Let k and m be positive integers.

DEFINITION 1.49. A Jacobi form of weight k and index m for L is a holomorphic function $\phi : \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ such that the function

$$\tilde{\phi}(Z) = \phi(\tau, z) e(m\omega), Z = \begin{pmatrix} \omega \\ z \\ \tau \end{pmatrix} \in \mathcal{H}_{n+2}$$

is a modular form of weight k with respect to O^J. Denote the $\mathbb{C}$-vector space of all such functions by $J_{k,m}(L)$.

It is also possible to define Jacobi forms with character in this way [**17**]. It is easy to check that $J_{k,m}(L) = J_{k,L(m)}$ for every positive-definite, even lattice $(L, (\cdot, \cdot))$ (see Definition 1.7). For example, for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, we have

$$\tilde{\phi}(\{A\}\langle Z\rangle) = \tilde{\phi}\begin{pmatrix} \omega - \frac{c(z,z)}{2(c\tau+d)} \\ \frac{z}{c\tau+d} \\ A\tau \end{pmatrix}$$

and, for every $\{x, y; r\}$ in $H(L)$, we have

$$\tilde{\phi}(\{x, y; r\}\langle Z\rangle) = \tilde{\phi}\begin{pmatrix} \omega + (x, z) + \frac{1}{2}(x, x)\tau + r \\ z + \tau x + y \\ \tau \end{pmatrix}.$$

Note that the integral Heisenberg group defined in Section 1.2.3 is isomorphic to the quotient of the integral Heisenberg group defined in this subsection modulo its center, in other words $H^L(\mathbb{Z}) \simeq H(L)/\{\{0, 0; n\} : n \in \mathbb{Z}\}$. The group $H^L(\mathbb{Z})$ is sometimes called the *reduced* Heisenberg group in the literature. The center of $H(L)$ acts trivially on Jacobi forms. Using the modularity condition (1.34) for $\tilde{\phi}$ and the above equations, it is possible to recover the modularity of ϕ as given by Definition 1.26, (1). We will use the notions discussed in this subsection in Chapter 4.

1.2.5. Examples of Jacobi forms. We have seen some examples of Jacobi forms of scalar index in Subsection 1.2.4.1. We list examples of Jacobi forms for some of the lattices in Example 1.6, as given in [**21**]. We remind the reader of the definitions of the Dedekind η-function (1.6) and the scalar Jacobi theta series (1.27).

EXAMPLE 1.50. For every n in $\mathbb{N}$, τ in $\mathfrak{H}$ and $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$, define

$$\vartheta_{\mathbb{Z}^n}(\tau, z) := \vartheta(\tau, z_1) \ldots \vartheta(\tau, z_n). \tag{1.35}$$

For $1 \leq n \leq 8$, the following function is a Jacobi form of weight $12 - n$ for the lattice D_n:

$$\psi_{12-n,D_n}(\tau, z) := \eta(\tau)^{24-3n} \vartheta_{\mathbb{Z}^n}(\tau, z).$$

When $n = 8$, this is a Jacobi form of singular weight for D_8. For $n \leq 7$, the function ψ_{12-n,D_n} is a cusp form. It is well-known that $D_3 = A_3$ and hence we also obtain a cusp form of weight 9 for A_3.

EXAMPLE 1.51. The function

$$\psi_{4,A_7}(\tau, z) = \vartheta(\tau, z_1) \ldots \vartheta(\tau, z_7) \vartheta(\tau, z_1 + \cdots + z_7)$$

is an element of J_{4,A_7} (we have written $z = (z_1, \ldots, z_7)$). Set

$$\Theta_{1,A_2}(\tau, z_1, z_2) := \frac{\vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 + z_2)}{\eta(\tau)}.$$

Then $\in J_{1,A_2}(v_\eta^8)$ and

$$\psi_{9,A_2} := \eta^{16}(\tau)\Theta_{1,A_2}(\tau, z_1, z_2)$$

is an element of S_{9,A_2}. We also have that

$$\psi_{6,2A_2}(\tau, z) = \eta^8(\tau)\Theta_{1,A_2}(\tau, z_1, z_2)\Theta_{1,A_2}(\tau, z_3, z_4) \in S_{6,2A_2}$$

(where $2A_2 = A_2 \oplus A_2$) and that

$$\psi_{3,3A_2}(\tau, z) = \Theta_{1,A_2}(\tau, z_1, z_2)\Theta_{1,A_2}(\tau, z_3, z_4)\Theta_{1,A_2}(\tau, z_5, z_6) \in J_{3,3A_2}.$$

Examples 1.50 and 1.51 are part of the theory of *theta blocks* developed in [**22**]. Here is another example from [**20**]:

EXAMPLE 1.52. Bearing in mind the modularity properties of theta series (1.19) and the fact that E_8 is a unimodular lattice, the theta series

$$\vartheta_{E_8}(\tau, z) = \sum_{r \in E_8} e\left(\tau \frac{(r, r)}{2} + (r, z)\right) \tag{1.36}$$

is an element of J_{4,E_8}. This is a Jacobi form of singular weight for E_8. Furthermore, fix an element x in E_8 and set $(x, x) = 2m$. Then the following function defined on $\mathfrak{H} \times \mathbb{C}$ is a Jacobi form of weight 4 and scalar index m:

$$\vartheta_{E_8,x}(\tau, z) := \vartheta_{E_8}(\tau, zx).$$

It has a Fourier expansion of the form

$$\vartheta_{E_8,x}(\tau, z) = 1 + \sum_{n \geq 0, l \in \mathbb{Z}} a(n, l) e(n\tau + lz),$$

where

$$a(n, l) = \#\{y \in E_8 : (y, y) = n \text{ and } (x, y) = l\}.$$

Note that the scalar Eisenstein series $E_{4,1,0}$ is equal to $\vartheta_{E_8, (\frac{1}{2}, \ldots, \frac{1}{2})}$.

In general, if $\underline{L}$ is an even, unimodular lattice, then

$$\vartheta_{\underline{L},0}(\tau, z) = \sum_{r \in L} e(\beta(r)\tau + \beta(r, z))$$

is a Jacobi form of singular weight $\frac{\mathrm{rk}(\underline{L})}{2}$ and index $\underline{L}$. The type of construction we encountered in the last example can be extended to arbitrary lattices in the following way: let $\phi \in J_{k,\underline{L}}$ and $\lambda \in L$; for a variable z in $\mathbb{C}$, the function $\phi(\tau, z\lambda)$ is a Jacobi form of weight k and scalar index $\beta(\lambda)$.

Lemma 3.34 implies that $W_t = W(s^{A_t})$ for every $t \parallel m$ and, conversely, that every operator $W(s^f)$ ($f^2 \equiv 1 \bmod 4m$) is equal to W_n for some $n \parallel m$ (the reader can consult the forward direction in the proof of the lemma for the precise recipe for finding n). The operators W_t are called *Atkin–Lehner involutions* in [**38**], because they play the role of Atkin–Lehner involutions for elliptic modular forms on the side of Jacobi forms. More precisely, the following holds:

$$\mathrm{tr}(T(l) \circ W_t, J_{k,m}) = \mathrm{tr}(T(l) \circ W_t, \mathfrak{M}^-_{2k-2}(m)),$$

It was shown in [**3**, §1.2] that the orthogonal groups of cyclic finite quadratic modules over number fields consist entirely of such operators W_t.

For the remainder of this section, we study *roots* of lattices and the corresponding elements in the discriminant module. An element λ in L is said to be *primitive* if, whenever $\lambda = m\mu$ for some m in $\mathbb{Z}$ and some μ in L, we have $m = \pm 1$. For every primitive element λ in L, define the *reflection map through* λ as

please state where you took this definition from.

$$s_\lambda(x) := x - \frac{\beta(x,\lambda)}{\beta(\lambda)}\lambda.$$

Definition 3.36 (Root). A root of L is a primitive element α of L such that reflection through α is an automorphism of L.

Proposition 3.37. *If α is a root of L, then the following hold:*

(i) *For every x in L, we have $\beta(\alpha) \mid \beta(x,\alpha)$.*
(ii) *Let $\langle\alpha\rangle$ denote the linear span of α inside L and let $\alpha^\perp$ denote the orthogonal complement of α with respect to β inside L. Then $\langle\alpha\rangle \cap \alpha^\perp = \{0\}$.*
(iii) *The automorphism s_α is an involution.*
(iv) *If γ is another root of L, then $s_\alpha(\gamma)$ is also a root.*

Proof. (*i*) Since s_α is an automorphism of L, we have

$$x - \frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha \in L$$

for all x in L. It follows that $\frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha \in L$ for all x in L. Suppose that $\frac{\beta(x,\alpha)}{\beta(\alpha)} = \frac{n}{m}$ for some integers m and n such that $(m,n) = 1$. Then $\alpha = \frac{m}{n}\lambda$ for some λ in L. Since $(m,n) = 1$, we have $\lambda/n = \mu \in L$ and therefore $\alpha = m\mu$. Since α is primitive, this implies that $m = \pm 1$ and hence $\frac{\beta(x,\alpha)}{\beta(\alpha)} \in \mathbb{Z}$.

(*ii*) Suppose that $\mu \in \langle\alpha\rangle \cap \alpha^\perp$ and that $\mu \neq 0$. Then $\mu = a\alpha$ for some a in $\mathbb{Z} \setminus \{0\}$ and $\beta(\mu,\alpha) = 0$, implying that $\beta(\alpha,\alpha) = 0$. For every x in L, we have $\beta(x,\mu) = a\beta(x,\alpha) = \beta(\alpha)n$ by the above, for some n in $\mathbb{Z}$. Since $\beta(\alpha) = 0$, it follows that $\beta(x,\alpha) = 0$ for every x in L and, since β is non-degenerate, this implies that $\alpha = 0$, which is a contradiction. Hence, $\langle\alpha\rangle \cap \alpha^\perp = \{0\}$.

(*iii*) We have

$$\begin{aligned} s_\alpha \circ s_\alpha(x) =& s_\alpha(x) - \frac{\beta(s_\alpha(x),\alpha)}{\beta(\alpha)}\alpha = x - \frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha - \frac{\beta\left(x - \frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha, \alpha\right)}{\beta(\alpha)}\alpha \\ =& x - \frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha - \frac{\beta(x,\alpha)}{\beta(\alpha)}\alpha + \frac{\beta(x,\alpha)}{\beta(\alpha)}\frac{\beta(\alpha,\alpha)}{\beta(\alpha)}\alpha = x \end{aligned}$$

and hence s_α is an involution.

(*iv*) Suppose that $s_\alpha(\gamma) = m\mu$, with m in $\mathbb{Z}$ and μ in L. Since s_α is an involution, we have

$$\gamma = s_\alpha \circ s_\alpha(\gamma) = ms_\alpha(\mu)$$

and, since γ is a root and therefore primitive, we have $m = \pm 1$. It follows that $s_\alpha(\gamma)$ is primitive.

For every x in L, we have

$$\begin{aligned}\beta\left(s_{s_\alpha(\gamma)}(x)\right) =&\beta\left(x - \frac{\beta(x, s_\alpha(\gamma))}{\beta(s_\alpha(\gamma))} s_\alpha(\gamma)\right)\\ =&\beta(x) - \frac{\beta(x, s_\alpha(\gamma))}{\beta(\gamma)}\beta(x, s_\alpha(\gamma)) + \frac{\beta(x, s_\alpha(\gamma))^2}{\beta(\gamma)^2}\beta(s_\alpha(\gamma)) = \beta(x)\end{aligned}$$

and therefore $\beta \circ s_{s_\alpha(\gamma)} = \beta$.

Suppose that $s_{s_\alpha(\gamma)}(x) = 0$. Then $x = \frac{\beta(x,s_\alpha(\gamma))}{\beta(\gamma)} s_\alpha(\gamma) \in L$. Since $s_\alpha(\gamma)$ is primitive, it follows from the same argument as in (*i*) that $\frac{\beta(x,s_\alpha(\gamma))}{\beta(\gamma)} \in \mathbb{Z}$. Write $x = ts_\alpha(\gamma)$ for simplicity, where $t := \frac{\beta(x,s_\alpha(\gamma))}{\beta(\gamma)}$. Then

$$x = ts_\alpha(\gamma) = \frac{\beta(ts_\alpha(\gamma), s_\alpha(\gamma))}{\beta(\gamma)} s_\alpha(\gamma) = 2ts_\alpha(\gamma).$$

Since $s_\alpha(\gamma) = 0 \iff \gamma = 0$, it follows that $t = 0$, i.e. that $x = 0$ and $s_{s_\alpha(\gamma)}$ is injective. To show that it is surjective, it suffices to prove that it is an involution. We have

$$\begin{aligned}s_{s_\alpha(\gamma)}\left(s_{s_\alpha(\gamma)}(x)\right) =&s_{s_\alpha(\gamma)}(x) - \frac{\beta\left(s_\alpha(\gamma), s_{s_\alpha(\gamma)}(x)\right)}{\beta(\gamma)} s_\alpha(\gamma)\\ =&x - \frac{\beta(x, s_\alpha(\gamma))}{\beta(\gamma)} s_\alpha(\gamma) - \frac{\beta(s_\alpha(\gamma), x)}{\beta(\gamma)} s_\alpha(\gamma)\\ &+ \frac{\beta(s_\alpha(\gamma), s_\alpha(\gamma))\beta(x, s_\alpha(\gamma))}{\beta(\gamma)^2} s_\alpha(\gamma) = x\end{aligned}$$

It follows that $s_{s_\alpha(\gamma)}$ is an automorphism of L and hence that $s_\alpha(\gamma)$ is a root. □

A consequence of Proposition 3.37, (*i*) is the following result from [**33**, §2]:

Proposition 3.38. *Let α be a root. Then $\beta(\alpha) \mid \mathrm{lev}(\underline{L})$ and $\alpha \in L \cap \beta(\alpha)L^\#$.*

Proof. Set $t := \beta(\alpha)$ for simplicity. Since $\beta(\alpha) \mid \beta(x, \alpha)$ for all x in L, we have $\beta(x, \alpha/t) \in \mathbb{Z}$ for all x in L, in other words $\alpha/t \in L^\#$. Furthermore, $\mathrm{lev}(\underline{L})/t = \mathrm{lev}(\underline{L})\beta(\alpha/t) \in \mathbb{Z}$ and therefore $t \mid \mathrm{lev}(\underline{L})$. □

The converse result, which we cite without proof, is the following:

Proposition 3.39. *If α in L is such that $\beta(\alpha) = t$ for some divisor t of $\mathrm{lev}(\underline{L})$ and $\alpha \in L \cap tL^\#$, then either α is a root or $\alpha/2$ is a root.*

A *root lattice* is a positive-definite, even lattice which is spanned by roots. The lattices in Example 1.6, (1), (3), (4) and (5) are root lattices.

On the side of discriminant modules of lattices, we have the following:

Definition 3.40 (Corresponding to roots). Let a in $L^\#/L$ be such that $\beta(a) \equiv 1/t \bmod \mathbb{Z}$, for some divisor t of $\mathrm{lev}(\underline{L})$. We say that *a corresponds to roots* if $N_a \mid t$ and if, provided there is an α in L such that $\alpha = ta$ and $\beta(\alpha) = t$, then α is a root.

Remark 1.14 implies that $N_a \mid t$ and $t \mid 2N_a$ if a corresponds to roots. It follows that either $\mathrm{lev}(a) = 2N_a$ or $\mathrm{lev}(a) = N_a$. In the former case we obtain that $ta/2 \in L$ and therefore a cannot correspond to a root $\alpha = ta$, since ta is not primitive. Thus, if a corresponds to a root α, then $N_a = \mathrm{lev}(a) = t$ necessarily. The converse also holds:

Proposition 3.41. *If a in $L^\#/L$ is such that $\beta(a) \equiv 1/N_a \bmod \mathbb{Z}$, then a corresponds to roots.*

Proof. If no $\alpha = N_a a$ exists such that $\beta(\alpha) = N_a$ and $\alpha \in L$, then we are done. Otherwise, Proposition 3.39 implies that either α or $\alpha/2$ is a root. Suppose that $\alpha = m\mu$ for some m in $\mathbb{Z}$ and some μ in L. Then $N_a = \beta(\alpha) = m^2\beta(\mu) \in \mathbb{Z}$, implying that $m \mid N_a$. It follows that $\mu = \alpha/m = (N_a/m)a = 0$ in $L^\#/L$ and therefore $N_a \mid (N_a/m)$. This implies that $m = \pm 1$ and hence α is primitive and a corresponds to α. □

Please do not, This is a fragment of a planned monograph which I gave confidentially to Fredrik (and asking him not to distribute).

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We introduce some additional notions from [**41**]:

Definition 3.42 (Admissible element). An element a of $L^\#/L$ is called admissible if

$$\beta(r, a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z} \tag{3.16}$$

for some integer $\varphi_a(r)$ for all r in $L^\#/L$.

We prove certain properties of admissible elements, some of which are listed in [**41**]:

Lemma 3.43. *If a is admissible, then the map $\varphi_a : L^\#/L \to \mathbb{Z}_{(\mathrm{lev}(a))}$, defined by*

$$\beta(r, a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z},$$

is a group homomorphism.

Proof. Since a is admissible, we have $\beta(r, a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z}$ for some integer $\varphi_a(r)$ by definition. Since $L^\#/L$ is an additive group, it follows that $\varphi_a(r_1 + r_2) = \varphi_a(r_1) + \varphi_a(r_2)$ for all r_1, r_2 in $L^\#/L$. We have $(\varphi_a(r) + n\,\mathrm{lev}(a))\beta(a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z}$ for every n in $\mathbb{Z}$ and therefore $\varphi_a(r)$ can be reduced modulo $\mathrm{lev}(a)$. □

Note that $\varphi_a(a) \equiv 2 \bmod \mathrm{lev}(a)$ for every admissible element a.

Proposition 3.44. *An element a is admissible if and only if $N_a = \mathrm{lev}(a)$ or $2N_a = \mathrm{lev}(a)$.*

Proof. Suppose that a is admissible. Then $\beta(r, a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z}$ for some integer $\varphi_a(r)$ for all $r \in L^\#/L$, implying that $\beta(r, \mathrm{lev}(a)a) \in \mathbb{Z}$ for all r in $L^\#/L$. Since β is non-degenerate, this implies that $\mathrm{lev}(a)a = 0$ in $L^\#/L$. It follows that $N_a \mid \mathrm{lev}(a)$. We remind the reader that $\mathrm{lev}(r) \mid 2N_r$ for every r in $L^\#/L$ and thus $N_a = \mathrm{lev}(a)$ or $2N_a = \mathrm{lev}(a)$.

For the converse statement, the two cases can be treated similarly and therefore we only include the proof when $N_a = \mathrm{lev}(a) = t$. Let r be an arbitrary element of $L^\#/L$. Since $ta \in L$, we have $t\beta(r, a) \in \mathbb{Z}$ and therefore $\beta(r, a) = t_r/t$ for some t_r in $\mathbb{Z}$. Since $\mathrm{lev}(a) = t$, we have $t\beta(a) = m$ for some m in $\mathbb{Z}$ which is coprime to t. It follows that there exist integers u and v such that $um + vt = 1$ and therefore $ut_r m + vt_r t = t_r$ and $t_r/t \equiv ut_r m/t \bmod \mathbb{Z}$. In other words, we have $\beta(r, a) \equiv \varphi_a(r)\beta(a) \bmod \mathbb{Z}$ for every r in $L^\#/L$, where $\varphi_a(r) \equiv u\,\mathrm{lev}(a)\beta(r, a) \bmod \mathrm{lev}(a)$, and hence a is admissible.

Note that, if we consider the case where $2N_a = \mathrm{lev}(a)$, then we also obtain that

$$\varphi_a(r) \equiv u\,\mathrm{lev}(a)\beta(r, a) \bmod \mathrm{lev}(a),$$

where u is such that $um + v\,\mathrm{lev}(a) = 1$ and m is such that $\beta(a) = m/\mathrm{lev}(a)$. □

For every a in $L^\#/L$, let $\langle a \rangle$ denote its $\mathbb{Z}$-span.

Proposition 3.45. *If a has odd order, then it is admissible if and only if $\beta|_{\langle a \rangle}$ is non-degenerate.*

Proof. Use the last proposition and instead prove that $N_a = \mathrm{lev}(a)$ if and only if $\beta|_{\langle a \rangle}$ is non-degenerate. Suppose that $N_a = \mathrm{lev}(a) = t$. Let ma in $\langle a \rangle$ be such that $\beta(\langle a \rangle, ma) = 0$. It follows that $\beta(a, ma) = 0$ and consequently that $2m$ is a multiple of

t. Since t is odd, this is equivalent to $t \mid m$, which implies that $ma = 0$ in $\langle a \rangle$. In other words, $\beta|_{\langle a \rangle}$ is non-degenerate.

Conversely, suppose that $\beta|_{\langle a \rangle}$ is non-degenerate. Then, if $\beta(\langle a \rangle, ma) = 0$ for some m in $\mathbb{Z}$, then $ma = 0$ and therefore $N_a \mid m$. In particular, it follows that $N_a \mid \operatorname{lev}(a)$ and, considering that $\operatorname{lev}(a) \mid N_a$ when N_a is odd (Remark 1.14), we obtain the desired equality. □

REMARK 3.46. The same result does not hold if N_a is even and equal to $\operatorname{lev}(a)$ because, in the forward direction, we can take $m = N_a/2$ and obtain that β is degenerate. However, the converse proof still works to show that, if $\beta|_{\langle a \rangle}$ is non-degenerate, then $2N_a = \operatorname{lev}(a)$ or $N_a = \operatorname{lev}(a)$ and therefore that a is admissible. It can also be shown that, if N_a is even and equal to $\operatorname{lev}(a)/2$, then $\beta|_{\langle a \rangle}$ is non-degenerate.

LEMMA 3.47. *An element a in $L^\#/L$ which corresponds to roots is admissible.*

PROOF. This is a consequence of Proposition 3.44 and of the fact that, if a corresponds to roots, then either $\operatorname{lev}(a) = 2N_a$ or $\operatorname{lev}(a) = N_a$. □

The converse of Lemma 3.47 need not hold, since it is not necessarily implied that $\beta(a) \equiv 1/\operatorname{lev}(a) \bmod \mathbb{Z}$ if a is admissible.

DEFINITION 3.48 (Reflection map). For every admissible element a in $L^\#/L$, define the reflection map through a as the function $s_a : L^\#/L \to L^\#/L$,

$$s_a(r) = r - \varphi_a(r)a.$$

PROPOSITION 3.49. *The map s_a is an element of $O(D_{\underline{L}})$ and an involution.*

PROOF. Let us first show that s_a is invariant under β:

$$\beta \circ s_a(r) = \beta(r - \varphi_a(r)a) = \beta(r) - \varphi_a(r)\beta(r, a) + \varphi_a^2(r)\beta(a) \equiv \beta(r) \bmod \mathbb{Z},$$

since the last two terms cancel out modulo $\mathbb{Z}$. It is clear that $s_a(\cdot)$ is a homomorphism, since $\varphi_a(\cdot)$ is one. To show that it is injective, assume that $s_a(r) = 0$. Then $r = \varphi_a(r)a$ and it follows that $\varphi_a(r) = \varphi_a(r)\varphi_a(a) = 2\varphi_a(r)$. Hence, $\varphi_a(r) = 0$ and therefore $r = 0$. To show that $s_a(\cdot)$ is surjective, it suffices to prove that it is an involution. We have

$$s_a \circ s_a(r) = r - \varphi_a(r)a - \varphi_a(r - \varphi_a(r)a)a = r - 2\varphi_a(r)a + \varphi_a(r)\varphi_a(a)a = r,$$

since $\varphi_a(a) = 2$. □

Note that, if $a = 0$, then s_a is the trivial automorphism. If a corresponds to roots, then we can take $u = 1$ in the proof of Proposition 3.44 and we obtain that

$$s_a(r) = r - \operatorname{lev}(a)\beta(r, a)a.$$

Furthermore, if a corresponds to a root α in L, then

$$s_a(r) = r - \frac{\beta(r, a)}{\beta(a)}a = s_\alpha(r).$$

The following result is a consequence of the above Proposition:

COROLLARY 3.50. *For every admissible a in $L^\#/L$, the operator $W(s_a)$ is Hermitian. Furthermore, Proposition 3.31 implies that $W(s_a)E_{k,\underline{L},r,\chi} = E_{k,\underline{L},s_a(r),\chi}$.*

In view of Theorem 3.11, the following holds:

COROLLARY 3.51. *Twisted Eisenstein series are common eigenforms of Hecke operators and operators arising from the action of reflection maps.*

hence $a = (a_1, a_2 2^t)$ with a_1 and a_2 odd and $0 \le t \le n$. In view of Proposition 3.53, we can assume that $(a_1, a_2) = 1$. We have $\beta(a) = (a_1^2 + 2^t a_1 a_2 + 2^{2t} a_2^2)/2^n$ and $\beta(r,a) = (2a_1 r_1 + 2^{t+1} a_2 r_2 + a_1 r_2 + 2^t a_2 r_1)/2^n$ for $r = (r_1, r_2)$ in $\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}$. Since $a_1^2 + 2^t a_1 a_2 + 2^{2t} a_2^2$ is coprime to 2^n, let u and v in $\mathbb{Z}$ be such that

$$u(a_1^2 + 2^t a_1 a_2 + 2^{2t} a_2^2) + v 2^n = 1. \tag{3.17}$$

Then, $\varphi_a(r) = u(2a_1 r_1 + 2^{t+1} a_2 r_2 + a_1 r_2 + 2^t a_2 r_1)$ and therefore

$$\begin{aligned} s_a(r) &= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} - u(2a_1 r_1 + 2^{t+1} a_2 r_2 + a_1 r_2 + 2^t a_2 r_1) \begin{pmatrix} a_1 \\ a_2 2^t \end{pmatrix} \\ &= \begin{pmatrix} r_1[1 - u(2a_1^2 + 2^t a_1 a_2)] - r_2 u(2^{t+1} a_1 a_2 + a_1^2) \\ r_2[1 - u(2^{2t+1} a_2^2 + 2^t a_1 a_2)] - r_1 u(2^{t+1} a_1 a_2 + 2^{2t} a_2^2)] \end{pmatrix}. \end{aligned}$$

Use (3.17) to write the last expression as:

$$s_a(r) = \begin{pmatrix} r_1 u(2^{2t} a_2^2 - a_1^2) - r_2 u(2^{t+1} a_1 a_2 + a_1^2) \\ -r_1 u(2^{t+1} a_1 a_2 + 2^{2t} a_2^2)] + r_2 u(a_1^2 - 2^{2t} a_2^2) \end{pmatrix}.$$

Admissible elements of the form $a = (a_1 2^t, a_2)$ gives rise to reflection maps

$$s_a(r) = \begin{pmatrix} r_1 u(a_2^2 - 2^{2t} a_1^2) + r_2[u(2^{2t} a_1^2 + 2a_2^2) - 2] \\ r_2 u(2^{2t} a_1^2 - a_2^2) + r_1[u(2^{2t+1} a_1^2 + a_2^2) - 2] \end{pmatrix},$$

where u and v are such that

$$u(2^{2t} a_1^2 + 2^t a_1 a_2 + a_2^2) + v 2^n = 1.$$

The case $\mathrm{lev}(a) = 2N_a$ cannot happen for this type of Jordan constituent, which follows from analyzing $\beta(a)$.

EXAMPLE 3.57. In the Jordan constituent $C_{2^n} = (\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}), \frac{xy}{2^n})$, every admissible element a of norm $N_a = 2^k$ is of the form $(m_1 2^{n-k}, m_2 2^{n-k+t})$, where $k \le n$ and $t \le k$ and m_1 and m_2 are odd. Then $\beta(a) = m_1 m_2 2^{n-2k+t}$ and, assuming that $\mathrm{lev}(\underline{L}) = N_a = 2^k$, we must have $n - k + t = 0$ by the same reasoning as above. It follows that $t = 0$ and $k = n$, i.e. that $a = (a_1, a_2)$ with a_1 and a_2 odd (we may also assume that they are coprime). We have $\beta(a) = a_1 a_2/2^n$ and $\beta(r,a) = (r_1 a_2 + r_2 a_1)/2^n$ for every $r = (r_1, r_2)$ in $\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}$. Hence, $\varphi_a(r) = u(r_1 a_2 + r_2 a_1)$, where u and v are such that $u a_1 a_2 + v 2^n = 1$, and

$$\begin{aligned} s_a(r) &= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} - u(r_1 a_2 + r_2 a_1) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} r_1(1 - u a_1 a_2) - r_2 u a_1^2 \\ r_2(1 - u a_1 a_2) - r_1 u a_2^2 \end{pmatrix} \\ &= \begin{pmatrix} -r_2 u a_1^2 \\ -r_1 u a_2^2 \end{pmatrix} = \begin{pmatrix} -r_2 a_1 a_2^{-1} \\ -r_1 a_1^{-1} a_2 \end{pmatrix} = \begin{pmatrix} -r_2 a_1 a_2^{-1} \\ -r_1 (a_1 a_2^{-1})^{-1} \end{pmatrix}, \end{aligned}$$

where x^{-1} denotes the inverse of x modulo 2^n. The case $\mathrm{lev}(a) = 2N_a$ cannot happen for this type of Jordan constituent either.

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3.3. Jacobi forms of index D_n and elliptic modular forms

In this section, we compute the Hecke eigenvalues of Jacobi cusp forms of weight k and index D_n for small values of k and odd n. We remind the reader of the definition of D_n given in Example 1.6, (3):

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

It is straight-forward to check that

$$D_n^\# = \left\{x : x \in \mathbb{Z}^n \text{ or } x \in \left(\tfrac{1}{2} + \mathbb{Z}\right)^n\right\}$$

and therefore

$$D_n^\#/D_n = \left\{0, e_n, \frac{e_1 + \cdots + e_n}{2}, \frac{e_1 + \cdots + e_{n-1} - e_n}{2}\right\},$$

where $\{e_i\}_i$ denotes the standard basis of $\mathbb{Z}^n$. Thus,

$$D_n^\#/D_n \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{if } n \text{ is odd and} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is even.} \end{cases}$$

Suppose that n is odd. Then the discriminant module associated with D_n is isomorphic to

$$\left(\mathbb{Z}/4\mathbb{Z}, r \mapsto \frac{nr^2}{8} \bmod \mathbb{Z}\right)$$

and $\operatorname{lev}(D_n) = 8$. It follows that D_n is stably isomorphic to D_m for every odd m and n such that $n \equiv m \bmod 8$ and, in view of Theorem 1.37, that $J_{k+\lceil\frac{n}{2}\rceil, D_n} \simeq J_{k+\lceil\frac{m}{2}\rceil, D_m}$ for such m and n. Hence, it suffices to consider $n = 1, 3, 5$ and 7 in this subsection.

In the following paragraphs, we introduce some building blocks for Jacobi forms. The Dedekind η-function was defined in (1.6). It is well-known that

$$\eta^3(\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) n q^{\frac{n^2}{8}} = \frac{1}{2}\sum_{n\in\mathbb{Z}} (-1)^n (2n+1) q^{\frac{(2n+1)^2}{8}}. \tag{3.18}$$

The scalar Jacobi theta series $\vartheta(\tau, z)$ was defined in (1.27) and the Jacobi theta series $\vartheta_{\mathbb{Z}^n}(\tau, z)$ was defined in (1.35). We remind the reader of the definition of the unimodular lattice E_8 given in Example 1.6, (5) and of that of the Jacobi theta series $\vartheta_{E_8}(\tau, z)$ given in (1.36). Let E_k ($k \geq 4$) denote the Eisenstein series of weight k for Γ,

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

and let E_2 denote the *quasi-modular* Eisenstein series of weight 2 for Γ,

$$E_2(\tau) := 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n.$$

The discriminant modular form, denoted by Δ, is a cusp form of weight 12 for Γ. It has the following Fourier expansion:

$$\Delta(\tau) = \sum_{n\geq 1} \tau(n) q^n,$$

where $\tau(n)$ is the Ramanujan tau function.

The differential operator $\partial : J_{k,\underline{L}} \longrightarrow J_{k+2,\underline{L}}$ is defined in [4] for every ϕ with theta expansion (1.20) as

$$\partial\phi(\tau, z) := \sum_{x\in L^\#/L} \left(q\frac{d}{dq} h_{\phi,x}(\tau)\right) \vartheta_{\underline{L},x}(\tau, z) - \frac{1}{12}\left(k - \frac{\operatorname{rk}(\underline{L})}{2}\right) E_2(\tau)\phi(\tau, z). \tag{3.19}$$

If $f(\tau) \in M_{k_1}(\Gamma)$ and $\phi(\tau, z) \in J_{k_2,\underline{L}}$, with Fourier expansions $\sum_{n\geq 0} a_f(n) q^n$ and (1.13), respectively, then it is easy to check that $f(\tau)\phi(\tau, z) \in J_{k_1+k_2,\underline{L}}$ and that

$$f(\tau)\phi(\tau, z) = \sum_{(D,r)\in \operatorname{supp}(\underline{L})} \left(\sum_{n=0}^{\lfloor -D\rfloor} C_\phi(D+n, r) a_f(n)\right) e((\beta(r) - D)\tau + \beta(r, z)). \tag{3.20}$$

For $n = 1, 3, 5$ and 7, let α_n denote the following embedding of D_n into E_8:

$$(x_1, \ldots, x_n) \mapsto (0, \ldots, 0, x_1, \ldots, x_n).$$

This map can be extended in a natural way to the underlying complex spaces. Its pull-back on spaces of Jacobi forms of weight k is the map $\alpha_n^* : J_{k,E_8} \to J_{k,D_n}$,

$$\alpha_n^*\phi(\tau, z) = \phi(\tau, \alpha_n(z)).$$

3.3.1. Generators and their Fourier expansions. The generators of the spaces J_{k,D_n} ($n = 1, 3, 5$ and 7) were listed in [**4**]. We compute their Fourier expansions in this subsection.

We remind the reader that the Jacobi forms ψ_{12-n,D_n} were defined in Example 1.50. Set

$$\begin{aligned} E_{4,D_n} &:= \alpha_n^*\vartheta_{E_8}, \\ E_{6,D_n} &:= \partial E_{4,D_n} \text{ and} \\ E_{8,D_n} &:= \partial E_{6,D_n}. \end{aligned}$$

Let σ_3 denote the following embedding of D_3 into $\mathbb{Z}^4$:

$$(x, y, z) \mapsto \frac{1}{2}(x + y - z, x - y + z, -x + y + z, -x - y - z)$$

and denote its pullback on spaces of Jacobi forms by σ_3^*.

THEOREM 3.58. *The following holds for $n = 1, 3, 5$ and 7:*

$$J_{2k+1,D_n} = M_*(1)\psi_{12-n,D_n}. \tag{3.21}$$

For $n = 1, 5$ and 7, we have

$$J_{2k,D_n} = M_*(1)E_{4,D_n} \oplus M_*(1)E_{6,D_n} \oplus M_*(1)E_{8,D_n} \tag{3.22}$$

and, lastly,

$$J_{2k,D_3} = M_*(1)E_{4,D_3} \oplus M_*(1)E_{6,D_3} \oplus M_*(1)\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}. \tag{3.23}$$

By definition,

$$\vartheta_{\mathbb{Z}^n}(\tau, z) = \sum_{r \in \mathbb{Z}^n} \left(\frac{-4}{r_1 \dots r_n}\right) e\left(\frac{r_1^2 + \cdots + r_n^2}{8}\tau + \frac{r_1 z_1 + \cdots + r_n z_n}{2}\right)$$

and therefore, using (3.18),

$$\begin{aligned} \psi_{12-n,D_n}(\tau, z) =& \frac{1}{2^{8-n}} \sum_{\substack{n_1, \dots, n_{8-n}, \\ m_1, \dots, m_n \in \mathbb{Z}}} \left(\frac{-4}{n_1 \dots n_{8-n} m_1 \dots m_n}\right) n_1 \dots n_{8-n} e\left(\frac{n_1^2 + \cdots + n_{8-n}^2 + m_1^2 + \cdots + m_n^2}{8}\tau + \frac{m_1 z_1 + \cdots + m_n z_n}{2}\right) \\ =& \sum_{r \in \left(\frac{1}{2} + \mathbb{Z}\right)^n} (-1)^{r_1 + \cdots + r_n - \frac{n}{2}} \sum_{x \in \left(\frac{1}{2} + \mathbb{Z}\right)^{8-n}} (-1)^{x_1 + \cdots + x_{8-n} - \frac{8-n}{2}} x_1 \dots x_{8-n} \\ & \times e\left(\left(\frac{(r,r)}{2} + \frac{(x,x)}{2}\right)\tau + (r, z)\right) \\ =& \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{\psi_{12-n,D_n}}(D, r) e\left(\left(\frac{(r,r)}{2} - D\right)\tau + (r, z)\right), \end{aligned}$$

where

$$C_{\psi_{12-n,D_n}}(D, r) = \begin{cases} 0, & \text{if } r \in \mathbb{Z}^n \text{ and} \\ \displaystyle\sum_{\substack{x \in \left(\frac{1}{2} + \mathbb{Z}\right)^{8-n} \\ -D = \frac{(x,x)}{2}}} (-1)^{r_1 + \cdots + r_n + x_1 + \cdots + x_{8-n}} x_1 \dots x_{8-n}, & \text{if } r \in \left(\frac{1}{2} + \mathbb{Z}\right)^n. \end{cases}$$

We have made the substitutions $x_i = \frac{n_i}{2}$ and $r_i = \frac{m_i}{2}$. The value of the expression $(-1)^{x_1+\cdots+x_n}x_1\ldots x_n$ does not change under the substitution $x_i := -x_i$ and therefore

$$C_{\psi_{12-n,D_n}}(D,r) = \begin{cases} 0, & \text{if } r\in\mathbb{Z}^n \text{ and} \\ 2^{8-n}\displaystyle\sum_{\substack{x\in\left(\frac{1}{2}+\mathbb{N}\right)^{8-n}\\ -D=\frac{(x,x)}{2}}} (-1)^{r_1+\cdots+r_n+x_1+\cdots+x_{8-n}}x_1\ldots x_{8-n}, & \text{if } r\in\left(\frac{1}{2}+\mathbb{Z}\right)^n. \end{cases}$$

We have

$$\begin{aligned} E_{4,D_n}(\tau,z) &= \sum_{r\in E_8} e\left(\tfrac{(r,r)}{2}\tau + r_{8-n+1}z_1+\cdots+r_8z_n\right) \\ &= \sum_{(r_{8-n+1},\ldots,r_8)\in\mathbb{Z}^n\cup\left(\frac{1}{2}+\mathbb{Z}\right)^n}\ \sum_{\substack{(r_1,\ldots,r_{8-n})\in\mathbb{R}^8\\ (r_1,\ldots,r_8)\in E_8}} e\left(\left(\frac{r_{8-n+1}^2+\cdots+r_8^2}{2}+\frac{r_1^2+\cdots+r_{8-n}^2}{2}\right)\tau\right) \\ &\quad\times e\left(r_{8-n+1}z_1+\cdots+r_8z_n\right) \\ &= \vartheta_{D_n,0}(\tau,z) + \sum_{\substack{r\in D_n^\#,\,D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{4,n}(D,r)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right), \end{aligned}$$

where

$$C_{4,n}(D,r) := \begin{cases} \#\left\{x\in\mathbb{Z}^{8-n} : -2D = x_1^2+\cdots+x_{8-n}^2\right\}, & r\in\mathbb{Z}^n \text{ and} \\ \#\left\{x\in\mathbb{Z}^{8-n} : -2D = x_1^2+x_1+\cdots+x_{8-n}^2+x_{8-n}+\frac{8-n}{4}\right\}, & r\in\left(\frac{1}{2}+\mathbb{Z}\right)^n. \end{cases}$$

Equations (3.19) and (3.20) imply that

$$\begin{aligned} E_{6,D_n}(\tau,z) = &\sum_{\substack{r\in D_n^\#,\,D\in\mathbb{Q}_{\le 0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} (-D)C_{4,n}(D,r)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right) \\ &- \frac{8-n}{24}\sum_{\substack{r\in D_n^\#,\,D\in\mathbb{Q}_{\le 0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{4,n}(D,r)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right) \\ &+ (8-n)\sum_{l\ge 1}\sigma_1(l)q^l \sum_{\substack{r\in D_n^\#,\,D\in\mathbb{Q}_{\le 0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{4,n}(D,r)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right) \\ = &\frac{n-8}{24}\vartheta_{D_n,0}(\tau,z) + \sum_{\substack{r\in D_n^\#,\,D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{6,n}(D,r)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right), \end{aligned}$$

where

$$C_{6,n}(D,r) := -\left(D+\frac{8-n}{24}\right)C_{4,n}(D,r) + (8-n)\sum_{l=1}^{\lfloor -D\rfloor} C_{4,n}(D+l,r)\sigma_1(l).$$

CHAPTER 4

Level raising operators

We define a generalization of the operators U_l and V_l from [**14**, §I.4] for Jacobi forms of lattice index and study some of their properties. Given the terminology on one hand and the connection between Jacobi forms and elliptic modular forms conjectured in [**1**, §6.1.1] on the other, the *level* of a Jacobi form should be the level of the lattice in its index. This is supported by results from [**31**], which state that the space of Jacobi newforms of weight k and scalar index 1 for $\Gamma_0(N)$ which is invariant with respect to the action of a certain Atkin–Lehner operator is isomorphic to the space of Jacobi newforms of weight k and scalar index N for Γ as modules over the Hecke algebra (for every odd, square-free N).

4.1. The U operators

These operators arise from *isometries* of lattices (see end of Subsection 1.2.2):

Definition 4.1. Let $\underline{L}_1$ and $\underline{L}_2$ bet two positive-definite, even lattices over $\mathbb{Z}$ such that there exists and isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Define a linear operator

$$U(\sigma) : J_{k,\underline{L}_2} \to \{\phi : \mathfrak{H} \times (L_1 \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C} : \phi \text{ is holomorphic}\}$$

as

$$U(\sigma)\phi(\tau, z_1) := \phi(\tau, \sigma(z_1)).$$

This operator satisfies the following:

Lemma 4.2. *Let σ be an isometry of $\underline{L}_1$ into $\underline{L}_2$. For every ϕ in $J_{k,\underline{L}_2}$, the function $U(\sigma)\phi$ is invariant with respect to the $|_{k,\underline{L}_1}$-action of $J^{\underline{L}_1}$.*

Proof. For every A in Γ, we have

$$\begin{aligned}(U(\sigma)\phi)|_{k,\underline{L}_1}A(\tau, z_1) =& U(\sigma)\phi\left(A\tau, \frac{z_1}{c\tau + d}\right)(c\tau + d)^{-k} e\left(\frac{-c\beta_1(z_1)}{c\tau + d}\right)\\ =&\phi\left(A\tau, \frac{\sigma(z_1)}{c\tau + d}\right)(c\tau + d)^{-k} e\left(\frac{-c\beta_2(\sigma(z_1))}{c\tau + d}\right)\\ =&\phi|_{k,\underline{L}_2}A(\tau, \sigma(z_1)) = \phi(\tau, \sigma(z_1)) = U(\sigma)\phi(\tau, z_1),\end{aligned}$$

since $\beta_2 \circ \sigma = \beta_1$ and ϕ is a Jacobi form of weight k and index $\underline{L}_2$.

On the other hand, for every (λ, μ) in $H^{\underline{L}_1}(\mathbb{Z})$, we have

$$\begin{aligned}(U(\sigma)\phi)|_{\underline{L}_1}(\lambda, \mu)(\tau, z_1) =& U(\sigma)\phi(\tau, z_1 + \lambda\tau + \mu) e(\tau\beta_1(\lambda) + \beta_1(\lambda, z_1))\\ =&\phi(\tau, \sigma(z_1) + \tau\sigma(\lambda) + \sigma(\mu))\, e(\tau\beta_2(\sigma(\lambda)) + \beta_2(\sigma(\lambda), \sigma(z_1)))\\ =&\phi|_{\underline{L}_2}(\sigma(\lambda), \sigma(\mu))(\tau, \sigma(z_1))\\ =&\phi(\tau, \sigma(z_1)) = U(\sigma)\phi(\tau, z_1),\end{aligned}$$

since $\beta_2 \circ \sigma = \beta_1$ and ϕ is a Jacobi form of index $\underline{L}_2$. It follows that $U(\sigma)\phi$ is invariant under the $|_{k,\underline{L}_1}$-action of $J^{\underline{L}_1}$, as claimed. □

We would like for $U(\sigma)\phi$ to be a Jacobi form of weight k and index $\underline{L}_1$. If ϕ in $J_{k,\underline{L}_2}$ has a Fourier expansion of the type

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then

$$U(\sigma)\phi(\tau, z_1) = \phi(\tau, \sigma(z_1)) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e\left(n\tau + \beta_2(r_2, \sigma(z_1))\right).$$

We need $\beta_2(r_2, \sigma(z_1)) = \beta_1(r_1, z_1)$ for some r_1 in $L_1^\#$ for every r_2 in $L_2^\#$ such that $c_\phi(n, r_2)$ is non-zero in order for $U(\sigma)\phi$ to have the correct Fourier expansion. One case in which this condition holds is when σ is surjective and we can make the change of variable $r' = \sigma^{-1}(r)$ in the above equation.

Assume that σ is surjective on $L_2^\#$. Then $\sigma : \sigma^{-1}(L_2^\#) \to L_2^\#$ is a $\mathbb{Z}$-module isomorphism and, furthermore,

$$\begin{aligned}
&\sigma^{-1}(L_2^\#) = \{r \in L_1 \otimes Q : \beta_2(x, \sigma(r)) \in \mathbb{Z} \text{ for all } x \text{ in } L_2\} \\
&\implies \text{for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_2(x, \sigma(r)) \in \mathbb{Z} \text{ for all } x \text{ in } \sigma(L_1) \\
&\iff \text{for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_1(\sigma^{-1}(x), r) \in \mathbb{Z} \text{ for all } x \text{ in } \sigma(L_1) \\
&\iff \text{for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_1(x', r) \in \mathbb{Z} \text{ for all } x' \text{ in } L_1 \\
&\implies \sigma^{-1}(L_2^\#) \subseteq L_1^\#.
\end{aligned}$$

This implies that $\sigma^{-1}(L_2^\#)$ is a $\mathbb{Q}$-submodule of $L_1^\#$ (since $L_1^\#$ is commutative and $\sigma^{-1}(L_2^\#)$ is an additive subgroup of $L_1^\#$) and hence that $\mathrm{rk}(\underline{L}_2) \leq \mathrm{rk}(\underline{L}_1)$. On the other hand, since $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ is an isometry and hence injective by definition, we also have that $\mathrm{rk}(\underline{L}_1) \leq \mathrm{rk}(\underline{L}_2)$. If follows that $\mathrm{rk}(\underline{L}_1) = \mathrm{rk}(\underline{L}_2)$, which is equivalent to the fact that $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-modules. Conversely, suppose that $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ satisfy $\mathrm{rk}(\underline{L}_1) = \mathrm{rk}(\underline{L}_2)$. Then every isometry σ of $\underline{L}_1$ into $\underline{L}_2$ is necessarily surjective as a map between $L_1 \otimes \mathbb{Q}$ and $L_2 \otimes \mathbb{Q}$ (since it is an injective linear map between $\mathbb{Q}$-modules of the same dimension). It follows that $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-modules and therefore it is invertible on $L_2^\#$. Hence, every isometry σ of $\underline{L}_1$ into $\underline{L}_2$ is invertible on $L_2^\#$ if and only if $\mathrm{rk}(\underline{L}_1) = \mathrm{rk}(\underline{L}_2)$ (if and only if $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$). As a consequence, the following holds:

with the same rank. said that

THEOREM 4.3. *Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be two positive-definite, even lattices over $\mathbb{Z}$, ~~such that $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ as modules over $\mathbb{Q}$ and~~ there exists an isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Then $U(\sigma)$ maps $J_{k,\underline{L}_2}$ to $J_{k,\underline{L}_1}$. Furthermore, if ϕ in $J_{k,\underline{L}_2}$ has a Fourier expansion of the type*

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then $U(\sigma)\phi$ has the following Fourier expansion:

$$U(\sigma)\phi(\tau, z_1) = \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ n\geq\beta_1(r_1), \sigma(r_1)\in L_2^\#}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)).$$

rank L_1 = rank L_2 ⟹ $L_1 \otimes \mathbb{Q} \cong L_2 \otimes \mathbb{Q}$ as vector space over $\mathbb{Q}$.

Proof. Lemma 4.2 implies that $U(\sigma)\phi$ transforms like a Jacobi form of weight k and index $\underline{L}_1$. In light of the discussion above regarding Fourier expansions, we have

$$\begin{aligned} U(\sigma)\phi(\tau, z_1) &= \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, \sigma(z_1))) \\ &= \sum_{\substack{n\in\mathbb{Z}, r_1\in\sigma^{-1}(L_2^\#) \\ n\geq\beta_1(r_1)}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)) \\ &= \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ n\geq\beta_1(r_1), \sigma(r_1)\in L_2^\#}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)), \end{aligned}$$

as claimed. □

Corollary 4.4. *Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be two positive-definite, even lattices over $\mathbb{Z}$, such* ~~*that $L_1\otimes\mathbb{Q} \simeq L_2\otimes\mathbb{Q}$ as modules over $\mathbb{Q}$ and*~~ *there exists an isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Then $U(\sigma)$ maps $S_{k,\underline{L}_2}$ to $S_{k,\underline{L}_1}$.*

Proof. If ϕ in $S_{k,\underline{L}_2}$ and has a Fourier expansion of the type

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n>\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then the above theorem implies that $U(\sigma)\phi$ has the following Fourier expansion:

$$U(\sigma)\phi(\tau, z_1) = \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ \sigma(r_1)\in L_2^\#, n\geq\beta_1(r_1)}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)).$$

If $n = \beta_1(r_1)$ in the above equation, then $n = \beta_2(\sigma(r_1))$ and hence $c_\phi(n, \sigma(r_1)) = 0$, since ϕ is a cusp form. It follows that $U(\sigma)\phi$ is also a cusp form. □

We will show that the $U(\cdot)$ operators preserve Eisenstein series in the following sections.

Remark 4.5. In Section 3.3, we encountered an example of an isometry of D_n into E_8 which is not surjective, but preserves Jacobi forms nonetheless:

$$\alpha_n : D_n \to E_8 : (x_1, \dots, x_n) \mapsto (0, \dots, 0, x_1, \dots, x_n).$$

This is due to the fact that, for every ϕ in J_{k,E_8}, we have

$$U(\alpha_n)\phi(\tau, z) = \alpha_n^*\phi(\tau, z) = \phi(\tau, \alpha_n(z)) = \sum_{\substack{n\in\mathbb{Z}, r\in E_8, \\ n\geq\frac{(r,r)}{2}}} c_\phi(n, r)e\,(n\tau + (r, \alpha_n(z)))$$

and note that

$$(r, \alpha_n(z)) = (\alpha_n((r_{8-n+1}, \dots, r_8)), \alpha_n(z)) = ((r_{8-n+1}, \dots, r_8), z)$$

and $(r_{8-n+1}, \dots, r_8) \in D_n^\#$ for every $r = (r_1, \dots, r_8)$ in E_8 (see Example 1.6). It follows that

$$\begin{aligned}
U(\alpha_n)\phi(\tau, z) &= \sum_{(r_{8-n+1},\dots,r_8)\in D_n^\#} \sum_{\substack{(r_1,\dots,r_{8-n})\\(r_1,\dots,r_8)\in E_8}} \sum_{\substack{n\in\mathbb{Z}\\ n\geq \frac{r_1^2+\cdots+r_8^2}{2}}} c_\phi(n,r)e\left(n\tau + ((r_{8-n+1},\dots,r_8), z)\right)\\
&= \sum_{\substack{(r_{8-n+1},\dots,r_8)\in D_n^\#, n\in\mathbb{Z}\\ n\geq \frac{r_{8-n+1}^2+\cdots+r_8^2}{2}}} \sum_{\substack{(r_1,\dots,r_{8-n})\\(r_1,\dots,r_8)\in E_8, n\geq\frac{r_1^2+\cdots+r_8^2}{2}}} c_\phi(n,r)e\left(n\tau + ((r_{8-n+1},\dots,r_8), z)\right)\\
&= \sum_{\substack{n\in\mathbb{Z}, s\in D_n^\#\\ n\geq\frac{(s,s)}{2}}} c_{U(\alpha_n)\phi}(n,s)e(n\tau + (s,z)),
\end{aligned}$$

where

$$c_{U(\alpha_n)\phi}(n,s) = \sum_{\substack{(x_1,\dots,x_{8-n})\\ (s_1,\dots,s_n,x_1,\dots,x_{8-n})\in E_8\\ n-\frac{s_1^2+\cdots+s_n^2}{2}\geq\frac{x_1^2+\cdots+x_{8-n}^2}{2}}} c_\phi\left(n, (s_1,\dots,s_n,x_1,\dots,x_{8-n})\right).$$

The operators $U(\cdot)$ *raise the level* of the index of the Jacobi form that they are applied to:

LEMMA 4.6. *If* $\underline{L}_1 = (L_1, \beta_1)$ *and* $\underline{L}_2 = (L_2, \beta_2)$ *are two positive-definite, even lattices over* $\mathbb{Z}$ *such that* ~~$L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ *as modules over* $\mathbb{Q}$~~ *and* σ *is an isometry of* $\underline{L}_1$ *into* $\underline{L}_2$, *then* $\operatorname{lev}(\underline{L}_2) \mid \operatorname{lev}(\underline{L}_1)$.

PROOF. By definition,

$$\begin{aligned}
\operatorname{lev}(\underline{L}_2) &= \min\{N \in \mathbb{N} : N\beta_2(r) \in \mathbb{Z} \text{ for all } r \text{ in } L_2^\#\}\\
&= \min\{N \in \mathbb{N} : N\beta_1(\sigma^{-1}(r)) \in \mathbb{Z} \text{ for all } r \text{ in } L_2^\#\}.
\end{aligned}$$

On the other hand, $\operatorname{lev}(\underline{L}_1)\beta_1(\sigma^{-1}(r)) \in \mathbb{Z}$ for all r in $L_2^\#$. Hence, $\operatorname{lev}(\underline{L}_2) \mid \operatorname{lev}(\underline{L}_1)$. □

EXAMPLE 4.7. The operator U_l defined in [**14**] arises from the following isometry of the lattice $(\mathbb{Z}, (x, y) \mapsto ml^2xy)$ into the lattice $(\mathbb{Z}, (x, y) \mapsto mxy)$:

$$\sigma_l : (\mathbb{Q}, (x, y) \mapsto ml^2xy) \to (\mathbb{Q}, (x, y) \mapsto mxy), \quad \sigma_l(x) = lx.$$

It raises the level by a factor of l^2.

Fix any two bases for $L_1 \otimes \mathbb{Q}$ and $L_2 \otimes \mathbb{Q}$, let G_1 and G_2 denote the Gram matrices of $\underline{L}_1$ and $\underline{L}_2$, respectively, and let M denote the matrix of σ with respect to these bases. Then

$$\beta_2 \circ \sigma = \beta_1 \iff M^tG_2M = G_1 \implies \det(\underline{L}_1) = \det(\underline{L}_2)\det(M)^2.$$

In other words, we have shown the following:

LEMMA 4.8. *If* $\underline{L}_1 = (L_1, \beta_1)$ *and* $\underline{L}_2 = (L_2, \beta_2)$ *are two positive-definite, even lattices over* $\mathbb{Z}$ *and* σ *is an isometry of* $\underline{L}_1$ *into* $\underline{L}_2$, *then* $\det(\underline{L}_1) = \det(\sigma)^2 \det(\underline{L}_2)$.

We remind the reader that $\operatorname{lev}(\underline{L})$ and $\det(\underline{L})$ have the same set of prime divisors for every fixed positive-definite, even lattice $\underline{L}$. It follows from this fact and from Lemmas 4.6 and 4.8 that, when $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$, the set of prime divisors of $\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}$ consists of the prime divisors of $\det(\sigma)$ which are not divisors of $\operatorname{lev}(\underline{L}_2)$, plus possibly some primes dividing $\operatorname{lev}(\underline{L}_2)$. Write

$$\operatorname{lev}(\underline{L}_1) \mid \delta \det(\underline{L}_1) \mid \operatorname{lev}(\underline{L}_1)^{\operatorname{rk}(\underline{L}_1)},$$

APPENDIX A

Tables of Hecke eigenvalues

This chapter contains the tables used in Section 3.3. The code which generates them is available at `https://github.com/am-github/eigenvalues-Dn`. The difficulty of computing the Hecke eigenvalues decreases as the rank of the lattice increases, since the Fourier coefficients of Jacobi forms of index D_n ($n = 1, 3, 5$ and 7) are linear functions of representation numbers of quadratic forms in $8 - n$ variables. It also increases with the weight for fixed n.

We remind the reader that

$$\lambda_\phi(l) = \frac{C_{T(l)\phi}(D, r)}{C_\phi(D, r)}$$

for every Hecke eigenform ϕ in $J_{k,\underline{L}}$ and every pair (D, r) in $\mathrm{supp}(\underline{L})$. The eigenvalues of Jacobi forms of weights 4, 6, 8, 10 and 12 and index D_n were computed for odd positive integers l using the pair $(-1, (0, \ldots, 0))$ in the support of D_n. The eigenvalues of Jacobi forms of weights $12 - n, 16 - n, 18 - n, 20 - n$ and $22 - n$ and index D_n were computed for odd positive integers l using the pair $\left(-\frac{n-1}{8}, \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right)$ in the support of D_n, unless $(l, n-1) > 1$. In the latter case, we replaced $-\frac{n-1}{8}$ with $-\frac{m}{8}$, where m is the smallest positive integer in the congruence class of $n - 1$ modulo 8 which is coprime to l.

TABLE A.1. Hecke eigenvalues of Jacobi forms of weights 4, 6, 8 and 10 and index D_1

l	$\lambda_{E_{4,D_1}}(l)$	$\lambda_{E_{6,D_1}}(l)$	$\lambda_{\psi_8}(l)$	$\lambda_{\psi_{10}}(l)$
1	1	1	1	1
3	244	19684	-1836	-4284
5	3126	1953126	3990	-1025850
7	16808	40353608	-433432	3225992
9	59293	387440173	1776573	-110787507
11	161052	2357947692	1619772	-753618228
13	371294	10604499374	-10878466	2541064526
15	762744	38445332184	-7325640	4394741400
17	1419858	118587876498	60569298	-5429742318
19	2476100	322687697780	-243131740	1487499860
21	4101152	794320419872	795781152	-13820149728
23	6436344	1801152661464	-606096456	-317091823464
25	9768751	3814699218751	-1204783025	289428769375

TABLE A.2. Hecke eigenvalues of Jacobi forms of weights 11 and 12 and index D_1

l	$\lambda_{\psi_{11}}(l)$	$\lambda_{\psi_{12}}(l)$	$\lambda_{\alpha_{12}}(l)$
1	1	1	1
3	-53028	71604	-128844
5	-5556930	-28693770	21640950
7	-44496424	-853202392	-768078808
9	1649707317	-5333220387	6140423133
11	6320674932	86731179612	-94724929188
13	-33124973098	-895323442786	-80621789794
15	294672884040	-2054588707080	-2788306561800
17	-722355252174	3257566804818	3052282930002
19	-1312620671860	23032467644420	-7920788351740
21	2359556371872	-61092704076768	98962345937952
23	3379752742152	146495714575224	-73845437470344
25	11805984696775	346495278609775	-8506441300625

TABLE A.3. Hecke eigenvalues of Jacobi forms of weights 15 and 17 and index D_1

l	$\lambda_{\psi_{15}}(l)$	$\lambda_{\psi_{17}}(l)$
1	1	1
3	-1016388	-19984212
5	-3341197410	42951708750
7	-51021361384	-16835358997576
9	-6592552918443	-218304667023003
11	-177413845094508	-7207832704992348
13	-264386643393418	-270053634881821882
15	3395952953155080	-858356053422255000
17	76811888571465906	-16275482960925966606
19	-147764402234885140	109087314160337984540

TABLE A.4. Hecke eigenvalues of Jacobi forms of weights 19 and 21 and index D_1

l	$\lambda_{\psi_{19}}(l)$	$\lambda_{\psi_{21}}(l)$
1	1	1
3	159933852	-735458292
5	-2838742578690	-16226178983250
7	-782281866962344	16050065775887864
9	-24452708083441803	-3511656253747419003
11	738502081164310452	-167429747630019631548
13	12249951000076215062	-1323691058888421756442
15	-454011035446304813880	11933677880707341609000
17	-5840692944055083371214	-496480799590583480551566
19	-10098306774778877636020	-11499782498758130928946180