How to construct explicitly Jacobi forms and vector valued modular forms, and how to to turn elliptic modular forms into Jacobi forms

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March 4, 2014

Automorphic Forms, Lie Algebras
and String Theory (March 3 - 6, 2014)
Laboratoire Paul Painlevé, Université Lille I

Vector valued modular forms are Jacobi forms

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Generating explicit formulas for Jacobi forms

- Generating explicit formulas for Jacobi forms is as easy as for (scalar valued) elliptic modular forms of integral weight.
- Jacobi forms are vector valued modular forms, but somehow *more complete* ones (Hecke theory, liftings, more algebraic and geometric structure).
- For computing vector valued modular forms it might often be easier to compute Jacobi forms.

Vector valued modular forms

Basic objects

- The nontrivial central extension $\mathrm{Mp}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z})$ by $\{\pm 1\}$:
  1. $\Big\{\big(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \sqrt{c\tau + d}\big) \mid \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})\Big\}$,
  2. $\big(A, v(\tau)\big) \cdot \big(B, w(\tau)\big) = \big(AB, v(B\tau)w(\tau)\big)$.
- A subgroup Γ of $\mathrm{Mp}(2,\mathbb{Z})$ of finite index, a Γ-left module V, with $\dim_{\mathbb{C}} V < \infty$, and a $k \in \frac{1}{2}\mathbb{Z}$.
- $M_k(V)$: the space of holomorphic $f : \mathbb{H} \to V$ such that
  1. $f|_k\alpha = \alpha.f$ for all α in Γ,
  2. for every β in $\mathrm{Mp}(2,\mathbb{Z})$ the function $\big|\{f|_k\beta\}(\tau)\big|$ is bounded to above in the half plane $\Im(\tau) \geq 1$.

Notations

- $\{f|_k(A,v)\}(\tau) = f(A\tau)/v(\tau)^{2k}$,
- $\alpha.f : \tau \mapsto \alpha.(f(\tau))$.

Jacobi forms

Basic objects

- An integral positive definite lattice $\underline{L} = (L, \beta)$ of rank n,
- A (half) integer $k \in \frac{1}{2}\mathbb{Z}$ s. th. $k \equiv n/2 \bmod \mathbb{Z}$,
- $J_{k+n/2,\underline{L}}$: the space of holomorphic functions $\phi : \mathbb{H} \times (\mathbb{C} \otimes L) \to \mathbb{C}$ s. th.
  1. $\phi|_{k+n/2,\underline{L}} A = \phi$ for all $A \in \mathrm{SL}(2, \mathbb{Z})$,
  2. $\phi(\tau, z + x\tau + y)\ \mathrm{e}(-\tau\beta(x) - \beta(x, z)) = \mathrm{e}(\beta(x + y))\,\phi(\tau, z)$ for all $x, y \in L$,
  3. The Fourier expansion of ϕ is of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ x\in L^\bullet \\ n\geq\beta(x)}} c(n, x)\ \mathrm{e}(n\tau + \beta(x, z))$$

($\beta(x) = \frac{1}{2}\beta(x, x)$ and $L^\bullet = L^\sharp$ if $\underline{L}$ even, otherwise the *shadow of* $\underline{L}$).

Jacobi's Jacobi forms

Jacobi's theta functions

$$\vartheta(\tau, z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}}$$
$$= q^{\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{n>0} \left(1 - q^n\right)\left(1 - q^n\zeta\right)\left(1 - q^n\zeta^{-1}\right)$$
$$\vartheta^*(\tau, z) = \sum_{r\in\mathbb{Z}} \left(\frac{12}{r}\right) q^{\frac{r^2}{24}} \zeta^{\frac{r}{2}} = \eta(\tau)\frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)} \quad \text{(Watson quintuple product)}$$
$$\vartheta(\tau, z) = 2\pi i \eta^3 z + O(z^3), \quad \vartheta^*(\tau, z) = \eta + O(z^2)$$

Proposition

- $\vartheta \in J_{1/2,\mathbb{Z}}(\varepsilon^3)$, $\vartheta^* \in J_{1/2,\mathbb{Z}(3)}(\varepsilon)$.

(Group of linear characters of $\mathrm{Mp}(2,\mathbb{Z})$ *equals* $\langle\varepsilon\rangle$*.)*

Examples of Jacobi forms

$$\Theta_2(\tau,x,y) = \vartheta^*(\tau,x)\vartheta^*(\tau,y) \in J_{1,\left[\begin{smallmatrix}3 & \\ & 3\end{smallmatrix}\right]}(\varepsilon^2)$$

$$\Theta_{2'}(\tau,x,y) = \sum_{\substack{a\in\mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \\ a\bar{a}\equiv 1 \bmod 12}} \psi(a)\, q^{\frac{a\bar{a}}{12}}\, e^{\frac{\pi i}{6}(x\bar{a}+ya)} \in J_{1,\left[\begin{smallmatrix}8 & 4\\ 4 & 8\end{smallmatrix}\right]}(\varepsilon^2)$$

$$\Theta_4(\tau,x,y) = \vartheta(\tau,x)\vartheta^*(\tau,x) \in J_{1,\left[\begin{smallmatrix}1 & \\ & 3\end{smallmatrix}\right]}(\varepsilon^4)$$

$$\Theta_6(\tau,x,y) = \vartheta(\tau,x)\vartheta(\tau,y) \in J_{1,\left[\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right]}(\varepsilon^6)$$

$$\Theta_8(\tau,x,y) = \vartheta(\tau,x)\vartheta(\tau,x+y)\vartheta(\tau,y)/\eta(\tau) \in J_{1,\left[\begin{smallmatrix}2 & 1\\ 1 & 2\end{smallmatrix}\right]}(\varepsilon^8)$$

$$\Theta_{10}(\tau,x,y) = \vartheta(\tau,x)\vartheta(\tau,x+y)\vartheta(\tau,x-y)\vartheta(\tau,y)/\eta(\tau)^2 \in J_{1,\left[\begin{smallmatrix}3 & \\ & 3\end{smallmatrix}\right]}(\varepsilon^{10})$$

$$\Theta_{14}(\tau,x,y) = \vartheta(\tau,x)\vartheta(\tau,y)\vartheta(\tau,x-y)\cdot$$
$$\cdot\,\vartheta(\tau,x+y)\vartheta(\tau,x+2y)\vartheta(\tau,2x+y)/\eta(\tau)^4 \in J_{1,\left[\begin{smallmatrix}8 & 4\\ 4 & 8\end{smallmatrix}\right]}(\varepsilon^{14}).$$

Modules of VVMFs and JFs

Proposition

$M_*(V) := \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} M_k(V)$ and $J_{*+n/2,\underline{L}} := \bigoplus_{k\in\frac{n}{2}+\mathbb{Z}} J_{k+n/2,\underline{L}}$ are free graded modules of finite rank over $M_* := \bigoplus_{k\in\mathbb{Z}_{\geq 0}} M_{2k}(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{C}[E_4, E_6]$.
The homogeneous elements of any basis over M_* have degree ≤ 12.
(Assumption: Γ contains a subgroup of finite index acting trivially on V.)

Example

For $\underline{L} = (\mathbb{Z}^2, (x,y) \mapsto x^t \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right] y)$, one has

$$J^{\mathrm{odd}}_{*+1,\underline{L}}(1) = M_*\, \vartheta(\tau, z_1)\vartheta(\tau, z_1+z_2)\vartheta(\tau, z_2)/\eta(\tau),$$
$$J^{\mathrm{ev.}}_{*+1,\underline{L}}(1) = M_*\, E_{4,\underline{L}} \oplus M_*\, E_{6,\underline{L}}.$$

Comments

JFs vs. VVMFs

- JFs are geometric objects:
  - For fixed τ a ϕ in $J_{k,\underline{L}}$ is a theta function on $(\mathbb{C} \otimes L)/\tau L + L$ $(L \subset \mathbb{C} \otimes L)$.
  - Jacobi forms have (sometimes) nice product expansions.
- JFs are arithmetic objects:
  - One can define Hecke operators acting on and L-functions of elements of $J_{k,\underline{L}}$ (Ali Ajouz, in progress).
  - For even $\underline{L}$ of odd rank n there should be (Hecke equivariant) liftings from $J_{k+n/2,\underline{L}}$ to $M_{2k-1}(\ell/4)$, where ℓ is the level of $\underline{L}$.
- JFs admit natural additional algebraic structures:
  We can multiply and differentiate (Rankin-Cohen brackets).
- JFs admit natural representation theoretic structures:
  Strong approximation holds for the Jacobi group over $\mathbb{Q}$.

Pure Γ-modules

Definition

For a congruence subgroup Γ of $\mathrm{Mp}(2,\mathbb{Z})$, a Γ-module V is called *pure* if

1. For some N the group $\Gamma(4N)^*$ acts trivially on V,
2. If $(1,-1)$ is in Γ, then it acts as a homothety.

Notations

$\Gamma(4N)^* = \{(A, j(A,\tau) \mid A \in \Gamma(4N)\}$, where $j(A,\tau)$ is the *Hecke* multiplier.

Remark

No. 2 is no restriction: $V = V^+ \oplus V^-$ as Γ-modules
($V^{\pm 1} = \pm 1 -$ eigenspace of $(1,-1)$), hence
$M_k(V) = M_k(V^+) \oplus M_k(V^-)$. But $M_k(V^\epsilon) = 0$ for $\epsilon \neq (-1)^{2k}$ (since
$f_k|(1,-1) = (-1)^{2k} f$).

Vector valued modular forms are Jacobi forms

Main Theorem (S.)

Let Γ *be a subgroup of* $\mathrm{Mp}(2,\mathbb{Z})$ *and* V *be a pure* Γ*-module. Then there exists an integral positive definite lattice* $\underline{L}$ *and a natural injection of graded* $M_*(\mathrm{SL}(2,\mathbb{Z}))$*-modules*

$$M_*(V) \to J_{*+n/2,\underline{L}},$$

where n *denotes the rank of* $\underline{L}$*.*

Remark

Very likely the "natural injection" of the graded M_*-modules is compatible with the action of double coset operators (provided there is such an action on $M_*(V)$).

JFs are VVMFs. I

Notations

- $\vartheta_{\underline{L},x}(\tau, z) = \sum_{\substack{r \in L^\sharp \\ r \equiv x \bmod L}} e(\tau\beta(r) + \beta(z, r))$ for $x \in L^\sharp$,
- $\Theta(\underline{L}) = \langle \vartheta_{\underline{L},x} \mid x \in L^\sharp/L \rangle$.

(Here $\underline{L}$ is assumed to be even. For odd $\underline{L}$ the space $\Theta(\underline{L})$ is a subspace of $\Theta(\underline{L}_{\text{ev.}})$.)

Proposition (Jacobi, Kloostermann, . . .)

$\Theta(\underline{L})$ *is an* $\mathrm{Mp}(2, \mathbb{Z})$*-module via the action* $(\alpha, \vartheta) \mapsto \vartheta|_{n/2,\underline{L}}\alpha^{-1}$.

Remark

The proposition holds also true for odd $\underline{L}$ (cf. *shadow theory of lattices*)

JFs are VVMFs. II

Proposition

There is a natural isomorphism

$$M_k(\Theta(\underline{L})) \xrightarrow{c} J_{k+n/2,\underline{L}}.$$

Proof.

Every f in $M_k(\Theta(\underline{L}))$ can be written with respect to the basis $\{\vartheta_{\underline{L},x}\}$ of $\Theta(\underline{L})$ as

$$f(\tau) = \sum_{x \in L^\sharp/L} h_x(\tau)\,\vartheta_{\underline{L},x}$$

. The natural isomorphism is

$$f \mapsto \text{``}(\tau, z) \mapsto \sum_{x \in L^\sharp/L} h_x(\tau)\,\vartheta_{\underline{L},x}(\tau, z)\text{''}.$$

Functorial principles

Proposition

For a Γ*-module* V *let* $V^{\uparrow}$ *denote the induced* $\mathrm{Mp}(2,\mathbb{Z})$*-module* $\mathbb{C}[\mathrm{Mp}(2,\mathbb{Z})] \otimes_{\mathbb{C}[\Gamma]} V$*. There is a natural isomorphism*

$$M_k(V) \xrightarrow{a} M_k(V^{\uparrow}).$$

Proof.

The isomorphism a is given by $f \mapsto \sum_{\alpha \in \mathrm{Mp}(2,\mathbb{Z})/\Gamma} \alpha \otimes f|_k \alpha^{-1}$. □

Proposition

If $V \xrightarrow{b} V'$ *is a* $\mathrm{Mp}(2,\mathbb{Z})$*-module homomorphism, then the application* $f \mapsto b \circ f$ *defines a map* $M_k(V) \xrightarrow{b_*} M_k(V')$*.*

Preparing the proof of the main theorem

The natural isomorphism of the main theorem

Given a Γ-module V, the "natural isomorphism" is the composition of the maps

$$M_k(V) \xrightarrow{a} M_k(V^{\uparrow}) \xrightarrow{b_*} M_k(\Theta(\underline{L})) \xrightarrow{c} J_{k+n/2,\underline{L}}$$

where $\underline{L}$ is a lattice such that there is an injection of $\mathrm{Mp}(2,\mathbb{Z})$-modules

$$V^{\uparrow} \xrightarrow{b} \Theta(\underline{L}).$$

Observation

For such an embedding b to exist, the $\mathrm{Mp}(2,\mathbb{Z})$-module $V^{\uparrow}$ (and hence the Γ-module V) must be pure since $\Theta(\underline{L})$ is pure.

The main lemma

Main Lemma

For every pure Γ*-module* V*, there is an even positive definite lattice* $\underline{L}$ *such that* $V^{\uparrow}$ *is isomorphic as* $\mathrm{Mp}(2,\mathbb{Z})$*-module to a submodule of* $\Theta(\underline{L})$*.*

Example

The $\mathrm{SL}(2,\mathbb{Z})$-module $\mathbb{C}[\mathrm{SL}(2,\mathbb{Z})] \otimes_{\mathbb{C}[\Gamma_0(2)]} \mathbb{C}(1)$ embeds into $\Theta(\underline{Z}^8_{\mathrm{ev.}})$, where $\underline{\mathbb{Z}}^8_{\mathrm{ev.}}$ is the sublattice of all eight-vectors in the standard lattice $\underline{\mathbb{Z}}^8$ whose sum of entries is even (i.e. equals D_8).

Weil representations. I

Proposition (Jacobi, . . . , Weil, 1967)

For every finite quadratic module $\underline{M} = (M, Q)$ there is a 'natural' action of $\mathrm{Mp}(2, \mathbb{Z})$ on $\mathbb{C}[M]$ (Weil representation $W(\underline{M})$ associated to $\underline{M}$).

Notations

A *finite quadratic module* is a pair (M, Q), where

- M is a finite abelian group, and
- $Q : M \to \mathbb{Q}/\mathbb{Z}$ is a quadratic form, i.e.
  1. $Q(ax) = a^2 Q(x)$ for all $x \in M$, $a \in \mathbb{Z}$,
  2. $Q(x, y) := Q(x + y) - Q(x) - Q(y)$ defines a non-degenerate bilinear form.

Weil representations. II

Example

The discriminant module $D_{\underline{L}} = (L^\sharp/L, x + L \mapsto \beta(x) + \mathbb{Z})$ of an even lattice $\underline{L}$.

Weil representation associated to $\underline{M} = (M, Q)$

- $\{([\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}], 1) \cdot \Psi\}(x) = \mathrm{e}(nQ(x))\,\Psi(x)$,
- $\{([\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}], \sqrt{\tau}) \cdot \Psi\}(x) = \frac{\sigma(\underline{M})}{\sqrt{|M|}} \sum_{y \in M} \mathrm{e}(Q(x, y))\,\Psi(y)$.

Proposition

$\Theta(\underline{L}) \cong W(D_{\underline{L}(-1)})$ as $\mathrm{Mp}(2, \mathbb{Z})$-modules.

Representations of $\mathrm{SL}(2, \mathbb{Z}_p)$

Theorem (Nobs-Wolfart, 1983)

Let q be a prime power. Every irreducible $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$-module V is isomorphic to a $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$-submodule of the Weil representation $W(\underline{M})$ of a suitable finite quadratic module $\underline{M}$.

Corollary

Every irreducible representation of $\mathrm{Mp}(2, \mathbb{Z})$ which factors through $\Gamma^*(4N)$ for some N is contained in the Weil representation of a suitable finite quadratic module.

Pure representations of $\mathrm{Mp}(2,\mathbb{Z})$

Theorem (S.)

Every pure $\mathrm{Mp}(2,\mathbb{Z})$*-module is contained in the Weil representation of a suitable finite quadratic module.*

Proposition

Let $\underline{L} = (L, \beta)$ *be an even positive definite lattice of even rank* n *whose level equals the exponent of the group* $L^\sharp/L$*. Then the dimension of the subspace of* $\mathrm{Mp}(2,\mathbb{Z})$*-invariant vectors in* $\Theta(N\underline{L})$ *tends to infinity as* N *grows.*

Lifting finite quadratic modules

Theorem (S.)

Every finite quadratic module is isomorphic to the discriminant module of an even positive definite lattice.

Remark

- The main point here is "positive definite".
- T.C. Wall (1965): Every finite quadratic module is a discriminant module of a (not necessarily positive) lattice.

Hint

The discriminant module of an $\underline{L}$ does only depend on the system of lattices $\mathbb{Z}_p \otimes L$, where p runs through the primes. Add unimodular lattices U_p to the $\mathbb{Z}_p \otimes L$ so that $U_p \perp (\mathbb{Z}_p \otimes L) = \mathbb{Z}_p \otimes L'$ for some lattice L', and so that the "oddity formula" implies that L' is positive definite.

MFs on $\Gamma_0(p)$ as JFs (p odd prime). I

- $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma_0(p)]} \mathrm{Res}_{\Gamma_0(p)} \mathbb{C}(1) \cong \mathbb{C}(1) \oplus \mathrm{St}$,
- $M_k(\Gamma_0(p)) \xrightarrow{a} M_k(\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma_0(p)]} \mathbb{C}(1)) \xrightarrow{b_*} M_k(\mathrm{SL}(2,\mathbb{Z})) \oplus M_k(\mathrm{St})$,
- $M_k(\Gamma_0(p)) = M_k(\mathrm{SL}(2,\mathbb{Z})) \oplus M_k^0(\Gamma_0(p))$.
- Let Q be the quaternion algebra which is ramified exactly at p and ∞. i.e. let $K = \mathbb{Q}(\sqrt{-p})$, and let $\ell = 1$ if $p \equiv 3 \bmod 4$, and, for $p \equiv 1 \bmod 4$, let ℓ be a prime such that ℓ is a quadratic non-residue module p and $\ell \equiv 3 \bmod 4$. Then $Q = K \oplus Kj$, where the multiplication is defined by the usual multiplication in the field K and by the rules $j^2 = -\ell$ and $aj = j\bar{a}$ ($a \in K$).
- Let $\mathfrak{o}$ be a maximal order containing $\mathbb{Z}_K + \mathbb{Z}_K j$, and set $\underline{L}_p := \big(\mathfrak{o}i, (x,y) \mapsto \mathrm{tr}(x\overline{y})/p\big)$.
- St is a submodule of $\Theta(\underline{L}_p)$.

MFs on $\Gamma_0(p)$ as JFs (p odd prime). II

Theorem

For any even integer k, the application

$$\lambda : f \mapsto \sum_{\substack{n \in \frac{1}{p}\mathbb{Z},\, n \geq 0, \\ r \in \mathfrak{o} \\ \mathrm{n}(r)/p \equiv -n \bmod \mathbb{Z}}} \big(a_f(n) - a_{\widehat{f}}(n)\big)\, q^{\mathrm{n}(r)/p+n}\, e\big(\operatorname{tr}(z\overline{r})/p\big) \qquad (1)$$

defines a map

$$\lambda : M_k\big(\Gamma_0(p)\big) \longrightarrow J_{k+2,\underline{L}_p}.$$

Here $f = \sum_{n\in\mathbb{Z}} a_f(n)q^n$ and $\widehat{f}(\tau) := f(-1/\tau)\tau^{-k} = \sum_{n\in\frac{1}{p}\mathbb{Z}} a_{\widehat{f}}(n)\, q^n$, and in (1) we set $a_f(n) = 0$ if n is not an integer. The kernel of λ equals $M_k\big(\mathrm{SL}(2,\mathbb{Z})\big)$, and its image consists of all Jacobi forms ϕ in $J_{k+2,\underline{L}_p}$ whose Fourier coefficients $c(n,r)$ have the property that the numbers $c(n+\mathrm{n}(r), r)$ depend only on $\mathrm{n}(r) \bmod \mathbb{Z}$.

MFs on $\Gamma_0(2)$ as JFs

Theorem

The application

$$f \mapsto \phi_f(\tau, z) = \sum_{\substack{n \in \frac{1}{2}\mathbb{Z},\ r \in (\mathbb{Z}^8_{ev.})^\sharp \\ r^2 \equiv -n \bmod \mathbb{Z}}} \big(a_f(n) + a_{\widehat{f}}(n)\big)\, q^{r^2+n}\, e\big(z \cdot r\big) \quad (z \in \mathbb{C}^8),$$

embeds $M_k(\Gamma_0(2))$ *into* $J_{k+4,\underline{Z}^8_{ev.}}$ *(k even). Here* $\widehat{f}(\tau) := f(-1/\tau)\tau^{-k} = \sum_{n \in \frac{1}{2}\mathbb{Z}} a_{\widehat{f}}(n)\, q^n$.
The image equals the subspace $J^+_{k+4,\underline{Z}^8_{ev.}}$ *of Jacobi forms which are* totally even *in the z-variable. In fact, one has*

$$J_{k+4,\underline{Z}^8_{ev.}} = J^+_{k+4,\underline{Z}^8_{ev.}} \oplus M_k(\mathrm{SL}(2,\mathbb{Z})) \cdot \prod_{j=1}^{8} \vartheta(\tau, z_j).$$

Methods to ‘compute’ JFs. I

Methods for generating Jacobi forms

1. Theta blocks,
2. Vector valued modular forms.
3. Taylor expansion around $z = 0$,
4. Modular symbols.

Methods to ‘compute’ JFs. II

Remarks

- *Theta blocks:* work not always, work nicely for small weights, appealing explicit formulas.
- *Vector valued modular forms:* yields some Jacobi forms in an easy ‘do it by hand way’ — provided the rank of the index is even and the level of the index is not too composite.
- *Taylor expansion:* works always, easy to implement, explicit closed formulas, becomes harder in terms of computational time for lattices of large level.
- *Modular symbols:* gives directly eigenforms, no need to generate whole spaces, closed appealing formulas — but currently works only for lattices of rank 1 (scalar index) due to lack of theory.

Finding generators for $J_{*+n/2,\underline{L}}$

- Compute the Hilbert-Poincaré series of $J_{*+n/2,\underline{L}}$:

$$\sum_{k\geq 0} \dim J_{k+n/2,\underline{L}} X^k = \frac{a_{12}X^{12} + \cdots + a_1 X + a_0}{(1-X^4)(1-X^6)}.$$

  Note:
  - a_k is the number of generators (of a homogeneous basis) of weight k.
  - Dimension formulas are known and easy to implement.
  - Typically, $a_n + \cdots + a_0$ equals $\frac{1}{2}\det(\underline{L})$.
- Try to find generators, using one or several of the described methods.

Theta blocks

- Given $\underline{L} = (L, \beta)$, find solutions of $\underline{\alpha} = (\alpha_1, \ldots, \alpha_N)$ ($\alpha_j : L \to \mathbb{Z}$ linear) of
$$\beta(x,x) = \alpha_1(x)^2 + \cdots + \alpha_N(x)^2.$$

- For any integer s, one has
$$\phi_{\underline{\alpha}}(\tau, z) := \theta\big(\tau, \alpha_1(z)\big) \cdots \theta\big(\tau, \alpha_N(z)\big)/\eta(\tau)^s \in J^!_{\frac{N}{2}-\frac{s}{2},\underline{L}}\big(\varepsilon^{3N-s}\big).$$

- $\phi_{\underline{\alpha}}$ is holomorphic at infinity iff
$$B\big(\alpha_1(x)\big) + \cdots + B\big(\alpha_N(x)\big) \geq \tfrac{s}{24}$$
for all x in $\mathbb{R} \otimes L$, where $B(t) = \frac{1}{2}(\mathtt{frac}(t) - \frac{1}{2})^2$.

Vector valued modular forms

- If $\Theta(\underline{L}) = V_1 \oplus \cdots \oplus V_r$ (as $\mathrm{Mp}(2,\mathbb{Z})$-modules), then
$$J_{k+n/2,\underline{L}} \cong M_k\big(\Theta(\underline{L})\big) \cong M_k(V_1) \oplus \cdots \oplus M_k(V_r).$$

- For any Γ in $\mathrm{Mp}(2,\mathbb{Z})$ and character χ on Γ, one has
$$M_k(\Gamma,\chi) \cong M_k(W_1) \oplus \cdots \oplus M_k(W_s)$$
if $\big(\mathbb{C}(\chi)\big)^{\uparrow} = W_1 \oplus \cdots \oplus W_r$. Accordingly,
$$M_k(\Gamma,\chi) = M_k(\Gamma,W_1) \oplus \cdots \oplus M_k(\Gamma,W_s).$$

- If $\mathrm{Res}_\Gamma\,\Theta(\underline{L})$ contains $\mathbb{C}(\chi)$, then $V_i \cong W_j$ for some (i,j)s, and for those
$$M_k(\Gamma,W_j) \hookrightarrow J_{k+n/2,\underline{L}}.$$

Taylor expansion around 0

- Let V in $\Theta(\underline{L})$ be a $\mathrm{Mp}(2,\mathbb{Z})$-submodule. Assume $U_0 : \vartheta \mapsto \vartheta(\tau, 0)$ is injective on V. Let $J^V_{k+n/2,\underline{L}}$ be the corresponding subspace of $J_{k+n/2,\underline{L}}$.
- Every ϕ in $J^V_{k+n/2,\underline{L}}$ can be written in the form $\phi = h \cdot \vartheta$, where h is a row vector of holomorphic functions in τ and ϑ a column vector whose entries form a basis of V.
- Set
$$W = \big(U_0\vartheta \;\, DU_0\vartheta \;\, \cdots D^{d-1}U_0\vartheta\big),$$
where $d = \dim V$ and $D = q\frac{d}{dq}$.
- The entries of hW are quasi-modular forms on $\mathrm{SL}(2,\mathbb{Z})$ (polynomials in E_2, E_4, E_6). 'Identify' the image Q of $\phi \mapsto hW$ in $\mathbb{C}[E_2, E_4, E_6]^d$.
- Then the application
$$F \mapsto FW^{-1}\vartheta$$
defines an isomorphism $Q \xrightarrow{\cong} J^V_{k+n/2,\underline{L}}$.

Period method

- For every positive integer m, one has Hecke-equivariant maps

$$J_{k,\underline{\mathbb{Z}}(2m)} \xrightarrow{S} M_{2k-2}(\Gamma_0(2m)) \xrightarrow{\lambda^{\pm 1},\cong} \mathrm{Hom}\left(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{C}[X,Y]_{k-2}\right)^{\pm 1}.$$

- Dualizing (and some algebraic manipulations) gives Hecke-equivariant maps

$$(\lambda^{\pm 1} \circ S)^* : \left(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \otimes \mathbb{C}[X,Y]_{k-2}\right)_{\Gamma_0(2m)} \to J_{k,\underline{\mathbb{Z}}(2m)}.$$

- Using the theory of theta lifts these maps can be made explicit. The resulting formulas are Jacobi theta series associated to ternary quadratic forms of signature $(2,1)$.

JFs over totally real number fields

Basic notions

- K is a totally real number field of degree d, $\mathfrak{o}$ its ring of integers, and $\mathfrak{d}$ its different.
- For every totally positive integral $\mathfrak{o}$-lattice $\underline{L}$, every weight k in $\frac{1}{2}\mathbb{Z}^d$ and every character of $\mathrm{Mp}(2,\mathfrak{o})$, one can define $J_{k,\underline{L}}(\chi)$ (*space of Jacobi forms over K*) (H. Boylan). ($\underline{L} = (L, \beta)$: L finitely generated torsion free $\mathfrak{o}$-module, $\beta(L, L) \subseteq \mathfrak{d}$)
- Jacobi forms over K are vector valued modular forms: $J_{k,\underline{L}}(\chi) \cong M_{k-(\frac{n}{2},\ldots,\frac{n}{2})}\big(\Theta(\underline{L})\big)$.
- There is a theory of finite quadratic $\mathfrak{o}$-modules and associated Weil representations of central two-fold extensions of $\mathrm{SL}(2,\mathfrak{o})$ (H. Boylan).
- $\Theta(\underline{L}) \cong W(D_{\underline{L}(-1)})$.

Hilbert modular forms as JFs?

Questions to solve

- Is every representation of $\mathrm{Mp}(2, \mathfrak{o})$ with finite image contained in a Weil representation associated to a finite quadratic module?
- Is every finite quadratic module over K isomorphic to a discriminant module of a totally positive definite lattice?

Answer to second question

No.
Consider the number field $K = \mathbb{Q}(\sqrt{17})$, where $\mathfrak{o} = \mathfrak{o}_K = \mathbb{Z}[\frac{1+\sqrt{17}}{2}]$ and $\mathfrak{d} = \sqrt{17}\mathfrak{o}$. Here $2\mathfrak{o} = \mathfrak{p}\mathfrak{p}'$ with $\mathfrak{p} = \pi\mathfrak{o}$ and $\mathfrak{p}' = \pi'\mathfrak{o}$, where $\pi = (5 + \sqrt{17})/2$ and $\pi' = (5 - \sqrt{17})/2$. The fqm. $\mathfrak{M} = (\mathfrak{o}/\pi\mathfrak{o}, x + \pi\mathfrak{o} \mapsto \frac{x^2}{\sqrt{17}\pi^2} + \mathfrak{d}^{-1})$ is not a discriminant module.
(Consider the Jordan decompositions of $\mathfrak{o}_{\mathfrak{p}'} \otimes L$ and $\mathfrak{o}_{\mathfrak{p}'} \otimes L$.)

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