On Siegel modular forms with respect to non-split symplectic groups

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In this talk, I will explain the following papers.

[1] H.Kitayama, "An explicit dimension formula for Siegel cusp forms with respect to the non-split symplectic groups", J. Math. Soc. Japan (2011).

[2] H.Kitayama, "On the graded ring of Siegel modular forms of degree two with respect to a non-split symplectic group", Internat. J. Math (2012).

[3] T.Ibukiyama and H.Kitayama, "Dimension formulas of paramodular forms of squarefree level and comparison with inner twist", in preparation.

(Today: only the case of indefinite quaternion algebra)

Our joint work proceeds as follows. (Details will be explained later.)

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Dimension formulas

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Dimension formulas

↓

Explicit construction of modular forms

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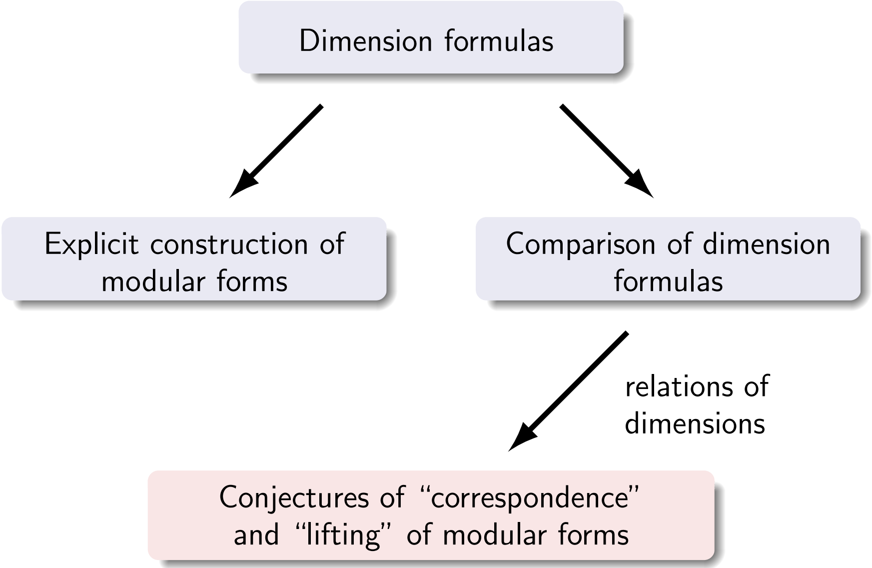

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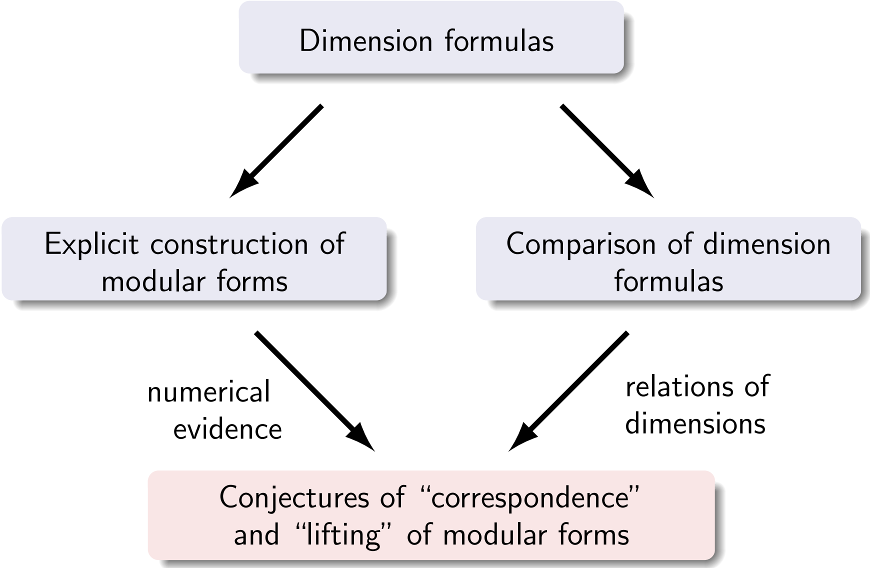

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§1 Explicit dimension formulas and comparison

$S_{k,j}(K(N))$

$S_{k,j}(U'(N))$

An explicit dimension formula of $S_{k,j}(K(N))$ (N: squarefree)

An explicit dimension formula of $S_{k,j}(U'(N))$ (N: squarefree)

Comparison

A relation of dimensions between $S_{k,j}(K(N))$ and $S_{k,j}(U'(N))$

$$Sp(2;\mathbb{R}) := \left\{ g \in GL(4;\mathbb{R}) \;\middle|\; g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} {}^t g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\},$$

$$\mathfrak{H}_2 := \{ Z \in M(2,\mathbb{C}) \mid {}^t Z = Z,\ \mathrm{Im}(Z) > 0 \}.$$

For

$\rho_{k,j} = \det^k \otimes \mathrm{Sym}_j$ $\quad (k, j \in \mathbb{Z}_{\geq 0})$: rep. of $GL(2;\mathbb{C})$

Γ: a discrete subgroup of $Sp(2;\mathbb{R})$ s.t. $\mathrm{vol}(\Gamma\backslash\mathfrak{H}_2) < \infty$,

Notation

$$S_{k,j}(\Gamma) = \{\textit{Siegel cusp forms of weight } \rho_{k,j} \textit{ w.r.t. } \Gamma\}$$

$$= \{ f : \mathfrak{H}_2 \to \mathbb{C}^{j+1} \ \textit{holomorphic} \quad \textit{s.t.}\ (1), (2)\}$$

(1) $f(\gamma\langle Z\rangle) = \rho_{k,j}(CZ + D) f(Z)$,

(2) $|\rho_{k,j}(\mathrm{Im}(Z)^{1/2}) f(Z)|_{\mathbb{C}^{j+1}}$ *is bounded on* $\mathfrak{H}_2$.

The first object of this work is paramoduler cusp forms.

Let N be a positive integer.

Definition (Paramodular group of level N)

$$K(N) := Sp(2;\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

Our starting point is to obtain an explicit dimension formula of $S_{k,j}(K(N))$ where N is square-free.

We can obtain explicit dimension formulas by using the method based on Selberg trace formula if $k \geq 5$.

Reference

S.Wakatsuki, "Dimension formulas for spaces of vector-valued Siegel cusp forms", J. Number theory (2012).

Procedure (Roughly speaking,)

Step 1: classify $K(N)$-conjugacy classes of $K(N)$.

Step 2: classify $K(N)$-conjugacy classes of families.
(≒ collect some $K(N)$-conjugacy classes.)

Step 3: sum up the contributions of each $K(N)$-conjugacy classes of families.

Theorem 1 ([3])

We assume N is a squarefree positive integer.
If $k \geq 5$ and $j \geq 0$ (even), then we have

$$\dim S_{k,j}(K(N)) = \sum_{i=1}^{12} H_i(k,j,N) + \sum_{i=1}^{10} I_i(k,j,N).$$

$$H_1(k,j,N) = \frac{(j+1)(k-2)(j+k-1)(j+2k-3)}{2^7 \cdot 3^3 \cdot 5} \cdot \prod_{p|N}(p^2+1)$$

$$H_2(k,j,N) = \frac{(j+k-1)(k-2)(-1)^k}{2^{8-\omega(N)} \cdot 3^2} \cdot \begin{cases} 11 & \cdots \quad \text{if } 2 \mid N, \\ 14 & \cdots \quad \text{if } 2 \nmid N \end{cases}$$

$$\vdots$$

$$I_{10}(k,j,N) = -\frac{[1,-1,0;3]_j}{2 \cdot 3} \cdot \prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right),$$

where $[1,-1,0;3]_j$ is the periodic function on j defined by mod 3.

The second object of this work is given as follows.

B: an indefinite quaternion algebra over $\mathbb{Q}$,

Definition

$$G_B := \left\{ g \in GL(2;B) \;\middle|\; g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Remark

1. G_B is a $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$. $\quad (G_B \overset{\text{fix}}{\hookrightarrow} Sp(2;\mathbb{R}))$
2. Any $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$ can be obtained as G_B for some indefinite quaternion algebra B over $\mathbb{Q}$.

- $\mathrm{disc}(B) = 1 \Longrightarrow G_B \simeq Sp(2;\mathbb{Q})$.
- $\mathrm{disc}(B) \neq 1 \Longrightarrow G_B \not\simeq Sp(2;\mathbb{Q})$.

We consider the case where $\mathrm{disc}(B) \neq 1$.

We consider the following discrete subgroup $U'(D)$ of $Sp(2;\mathbb{R})$.

Let

- D: the discriminant of B,
- $\mathfrak{O}$: the maximal order of B (unique up to conjugation)
- $\mathfrak{A}$: the maximal two-sided ideal of $\mathfrak{O}$ determined by

$$\begin{cases} \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \pi \mathfrak{O}_p \cdots \text{ if } p \mid D, \\ \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_p \cdots \text{ if } p \nmid D, \end{cases}$$

where π is a prime element of $\mathfrak{O}_p$.

Definition

$$U'(D) := G_B \cap \begin{pmatrix} \mathfrak{O} & \mathfrak{A}^{-1} \\ \mathfrak{A} & \mathfrak{O} \end{pmatrix} \subset Sp(2;\mathbb{R}).$$

We consider $S_{k,j}(U'(D))$, the space of Siegel cusp forms of weight $\rho_{k,j}$ with respect to $U'(D)$.

An explicit dimension formula of $S_{k,j}(U'(N))$ is as follows.
($\omega(N)$ is the number of prime divisors of N.)

Theorem 2 ([1])

We assume N is a square-free positive integer and $\omega(N)$ is even.
If $k \geq 5$ and $j \geq 0$(even), then we have

$$\dim S_{k,j}(U'(N)) = \sum_{i=1}^{12} H'_i(k,j,N) + \sum_{i=1}^{10} I'_i(k,j.N).$$

$$H'_1(k,j,N) = \frac{(j+1)(k-2)(j+k-1)(j+2k-3)}{2^7 \cdot 3^3 \cdot 5} \cdot \prod_{p|N}(p^2-1)$$

$$H'_2(k,j,N) = \frac{(j+k-1)(k-2)(-1)^k}{2^7 \cdot 3^2} \cdot \begin{cases} 3 & \text{if } N = 2 \\ 0 & \text{if } N \neq 2 \end{cases}$$

$$\vdots$$

Comparison

We want to compare

$$\boxed{\text{dimension formula for } S_{k,j}(K(N))} \qquad \text{(Theorem 1)}$$

with

$$\boxed{\text{dimension formulas for } S_{k,j}(U'(N))} \qquad \text{(Theorem 2).}$$

We compare

$$\sum_{M|N} \left((-2)^{\omega(M)} \dim S_{k,j}(K(N/M)) \right) \quad \text{with} \quad \dim S_{k,j}(U'(N)).$$

($\omega(M)$ is the number of prime divisors of M)

We can obtain the following result:

$$\sum_{M|N}\left((-2)^{\omega(M)}\dim S_{k,j}(K(N/M))\right)-\dim S_{k,j}(U'(N))$$

$$=\cdots$$

$$\vdots$$

$$=\cdots$$

$$=-\sum_{\substack{M|N\\ \omega(M)=\text{odd}}}\dim S^{new}_{j+2}(\Gamma_0^{(1)}(M))\times\dim S^{new}_{2k+j-2}(\Gamma_0^{(1)}(N/M)).$$

If $j=0$, then $\dim S^{new}_{j+2}(\Gamma_0^{(1)}(M))$ should be replaces by

$$\dim S^{new}_{j+2}(\Gamma_0^{(1)}(M))+1.$$

We obtained the following theorem.

Theorem 3 ([3])

We assume that N is a squarefree positive integer and $\omega(N)$ is even. If $k \geq 5$ and $j \geq 0$, then we have

$$\sum_{M|N} (-1)^{\omega(M)} 2^{\omega(M)} \dim S_{k,j}(K(N/M))$$
$$= \dim S_{k,j}(U'(N))$$
$$- \sum_{\substack{M \mid N, \\ \omega(M) = odd}} (\dim S_{j+2}^{new}(\Gamma_0^{(1)}(M)) + \delta_{j,0}) \times \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)).$$

- This theorem makes us expect a correspondence between $S_{k,j}(K(N))$ and $S_{k,j}(U'(N))$, which will be discussed in §2.
- We have the same result as Theorem 3 also for the case where $\omega(N)$ is odd, but we omit it in this talk.

§2 Conjectures on correspondence and lifting

Theorem 3 (obtained in §1)

A relation of dimensions between $S_{k,j}(K(N))$ and $S_{k,j}(U'(N))$

- Definition of paramodular newforms
- Conjectural dimension formula of $S_{k,j}^{new}(K(N))$

Theorem 3'(New version of Theorem 3)

$\dim S_{k,j}^{new}(K(N)) = \cdots\cdots$

(The RHS suggests how we should define $S_{k,j}^{new}(U'(N))$.)

Conjectures

"Correspondence" and "Lifting"

First, we review the definition of paramodular newforms of level p.

Reference

T. Ibukiyama, "On relations of dimensions of automorphic forms of $Sp(2,\mathbb{R})$ and its compact twist $Sp(2)$ (I)", 1985.

For a prime number p, $Sp(2;\mathbb{Q}_p)$ has three standard maximal compact subgroups.

The global discrete subgroups corresponding to them are

$$K(p),\quad Sp(2;\mathbb{Z}),\quad U_p^{-1}Sp(2;\mathbb{Z})U_p,$$

where $U_p = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & p & & \\ p & & & \end{bmatrix}$.

- There are no inclusion relations among these three groups.
- But we can define a mapping to $S_{k,j}(K(p))$ by taking traces.

Put

$$\Gamma_0'(p) := K(p) \cap Sp(2;\mathbb{Z}), \qquad \Gamma_0''(p) := K(p) \cap U_p^{-1} Sp(2;\mathbb{Z}) U_p.$$

We define the following two operators.

$$\theta_p : S_{k,j}(Sp(2;\mathbb{Z})) \longrightarrow S_{k,j}(K(p))$$

by

$$F|\theta_p := \sum_{g \in \Gamma'(p) \backslash K(p)} F|_{k,j}[g].$$

$$\theta_p' : S_{k,j}(U_p^{-1} Sp(2;\mathbb{Z}) U_p) \longrightarrow S_{k,j}(K(p))$$

by

$$F|\theta_p' := \sum_{g \in \Gamma''(p) \backslash K(p)} F|_{k,j}[g].$$

All the cusp forms in the images of θ_p and θ_p' should be considered "oldforms".

Definition (Oldforms)

$$S_{k,j}^{old}(K(p)) := S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p + S_{k,j}(U_p^{-1}Sp(2;\mathbb{Z})U_p)|\theta'_p$$
$$= S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p + S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p U_p$$

(Note that the sum is NOT the direct sum in general.)

The space of newforms is defined as the orthogonal complement of $S_{k,j}^{old}(K(p))$ in $S_{k,j}(K(p))$ with respect to the usual Peterson inner product.

Definition (Newforms)

$$S_{k,j}^{new}(K(p)) := S_{k,j}^{old}(K(p))^{\perp}.$$

(Note that our definition does NOT mean that newforms are not lift.)

We want to consider $\dim S_{k,j}^{new}(K(N))$.

If $j = 0$, there is a Saito-Kurokawa lifting

$$\mathrm{SK} : S_{2k-2}(SL_2(\mathbb{Z})) \longrightarrow S_{k,0}(Sp(2;\mathbb{Z})). \qquad \text{(injective)}$$

(if k is even.)

Lemma (Roberts-Schmidt (2006) Theorem 6.4)

Let $F \in S_{k,0}(Sp(2;\mathbb{Z}))$. We assume that Ramanujan Conjecture holds. Then

$$F \text{ is in the image of SK} \quad \Longleftrightarrow \quad F|\theta_p = F|\theta_p U_p.$$

Then, by this lemma, we can expect that the dimension of $S_{k,j}^{old}(K(p))$ should be as follows:

$$\dim S_{k,j}^{old}(K(p)) = \dim S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p + \dim S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p U_p - \delta_{j,0} \dim \mathrm{Image}(SK)$$

$$\begin{aligned}\dim S_{k,j}^{old}(K(p)) &= \dim S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p + \dim S_{k,j}(Sp(2;\mathbb{Z}))|\theta_p U_p \\ &\quad - \delta_{j,0} \dim \mathrm{Image}(SK) \\ &= 2 \dim S_{k,j}(Sp(2;\mathbb{Z})) \\ &\quad - \delta_{j,0} \cdot \begin{cases} \dim S_{2k-2}(SL_2(\mathbb{Z})) & \cdots \quad \text{if } k \text{ is even} \\ 0 & \cdots \quad \text{if } k \text{ is odd} \end{cases} \\ &= 2 \dim S_{k,j}(Sp(2;\mathbb{Z})) \\ &\quad - \delta_{j,0} \cdot \dim S_{2k-2}^{new,-}(SL_2(\mathbb{Z})).\end{aligned}$$

Hence we have

$$\begin{aligned}\dim S_{k,j}^{new}(K(p)) &= \dim S_{k,j}(K(p)) - \dim S_{k,j}^{old}(K(p)) \\ &= \dim S_{k,j}(K(p)) - 2 \dim S_{k,j}(Sp(2;\mathbb{Z})) \\ &\quad + \delta_{j,0} \cdot \dim S_{2k-2}^{new,-}(SL_2(\mathbb{Z})).\end{aligned}$$

Similarly, we consider the squarefree level case.

Let

N : a squarefree positive integer, p : a prime divisor of N.

Definition

We define

$$\theta_p : S_{k,j}(K(N/p)) \longrightarrow S_{k,j}(K(N))$$

by

$$F|\theta_p := \sum_{g \in K(N/p) \cap K(N) \backslash K(N)} F|_{k,j}[g].$$

Definition

$$S_{k,j}^{old}(K(N)) := \sum_{p|N} S_{k,j}(K(N/p))|\theta_p + \sum_{p|N} S_{k,j}(K(N/p))|\theta_p U_N$$

$$S_{k,j}^{new}(K(N)) := S_{k,j}^{old}(K(N))^{\perp}$$

By careful considerations, we can prove the following formula under some assumptions.

Conjectural dimension formula

Let N be a squarefree positive integer. Then we have

$$\dim S_{k,j}^{new}(K(N)) = \sum_{M|N} (-1)^{\omega(M)} 2^{\omega(M)} \dim S_{k,j}(K(N/M)) - \delta_{j,0} \cdot \sum_{\substack{M|N \\ M \neq 1}} (-1)^{\omega(M)} \dim S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)),$$

where $S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(m)) = \left\{ f \in S_{2k-2}^{new}(\Gamma_0^{(1)}(m)) \mid u_m f = \mp(-1)^k f \right\}$.

(u_m is the Atkin-Lehner involution defined by the action of $\begin{bmatrix} 0 & -1 \\ m & 0 \end{bmatrix}$.)

Then we obtain the following theorem.

Theorem 3' (New version of Theorem 3)

Let N be a squarefree positive integer. Let k be an integer ≥ 5 and $j > 0$ (even). If $\omega(N)$ is even, then we have

$$\begin{aligned}\dim S^{new}_{k,j}(K(N)) = &\dim S_{k,j}(U'(N)) \\ &- \sum_{\substack{M|N,\\ \omega(M)=odd}} \dim S^{new}_2(\Gamma_0^{(1)}(M)) \times \dim S^{new}_{2k-2}(\Gamma_0^{(1)}(N/M)) \\ &- \delta_{j,0} \cdot \sum_{\substack{M|N,\\ \omega(M)=odd}} \dim S^{new,+}_{2k-2}(\Gamma_0^{(1)}(N/M)) \\ &- \delta_{j,0} \cdot \sum_{\substack{M|N,\\ \omega(M)=even}} \dim S^{new,-}_{2k-2}(\Gamma_0^{(1)}(N/M)).\end{aligned}$$

The RHS suggests how we should define newforms for $S_{k,j}(U'(N))$. And we can read how the lifting should be in $S_{k,j}(U'(N))$.

Conjecture 1

(i) If $j = 0$, the spaces $S_{k,0}(U'(N))$ have liftings from

$$\begin{cases} S_2^{new}(\Gamma_0^{(1)}(M)) \times S_{2k-2}^{new}(\Gamma_0^{(1)}(N/M)) & : \quad M \mid N,\ \omega(M) = odd, \\ S_{2k-2}^{new,+}(\Gamma_0^{(1)}(N/M)) & : \quad M \mid N,\ \omega(M) = odd, \\ S_{2k-2}^{new,-}(\Gamma_0^{(1)}(N/M)) & : \quad M \mid N,\ \omega(M) = even \end{cases}$$

(ii) If $j \geq 2$, the spaces $S_{k,j}(U'(N))$ have liftings from

$$\left\{ S_{j+2}^{new}(\Gamma_0^{(1)}(M)) \times S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) \quad : \quad M \mid N,\ \omega(M) = odd \right.$$

- All the above lifting maps are injective.
- For $g \in S_{j+2}^{new}(\Gamma_0^{(1)}(M))$ and $f \in S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M))$,

$$L(s, \iota(f,g), Sp) = L(s,f)L(s-k+2,g).$$

- For $f \in S_{2k-2}^{new,\pm}(\Gamma_0(M))$,

$$L(s, \iota(f), Sp) = \zeta(s-k+1)\zeta(s-k+2)L(s,f).$$

(up to bad Euler factors)

If we define $S^{new}_{k,j}(U'(N))$ by the orthogonal complement of all our conjectural liftings in $S_{k,j}(U'(N))$,

then Theorem 3' is

$$\dim S^{new}_{k,j}(K(N)) = \dim S^{new}_{k,j}(U'(N))$$

We propose the following conjecture.

Conjecture 2

Let N be a squarefree positive integer. Then we have isomorphisms

$$S^{new}_{k,j}(K(N)) \xrightarrow{\sim} S^{new}_{k,j}(U'(N))$$

which preserve L-functions.

§3 Numerical evidence

Theorem ([2])

Explicit construction of the graded ring $\bigoplus_{k=0}^{\infty} M_k(U'(6))$

↓

Experiments of Hecke operator on $S_k(U'(6))$

↓

Numerical evidence

- examples of Conjecture 1 for small k
- examples of coincidence of L-functions in Conjecture 2

Theorem ([2])

$$\bigoplus_{k=0}^{\infty} M_k(U'(6)) = \mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{15}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\chi_{15}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6],$$

where

E_2, E_4, E_6 : the Eisenstein series of weight 2, 4, 6,

$\chi_{5a}, \chi_{5b}, \chi_{15}$: Siegel cusp forms of weight 5, 5, 15,

and

E_2, E_4, χ_{5a} and E_6 are algebraically independent over $\mathbb{C}$,

χ_{5b}^2 and χ_{15}^2 belong to $\mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$.

- H.Kitayama, "On the graded ring of Siegel modular forms of degree two with respect to a non-split symplectic group", Internat.J.Math (2012).
- Y.Hirai, "On Eisenstein series on quaternion unitary groups of degree 2", J.Math.Soc.Japan (1999).

For simplicity, we write $\Gamma := U'(N)$ and $S_k(\Gamma) = S_{k,0}(\Gamma)$.

Definition

For $m \in \mathbb{N}$ and $f \in S_k(\Gamma)$, we define

$$\left(T_k(m)f\right)(Z) := m^{2k-3} \sum_{\sigma \in \Gamma \backslash S_m} \det(CZ+D)^{-k} f(\sigma\langle Z \rangle)$$

where

$$S_m := \left\{ g \in M(2;B) \,\middle|\, g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \cap \begin{pmatrix} \mathfrak{O} & \mathfrak{A}^{-1} \\ \mathfrak{A} & \mathfrak{O} \end{pmatrix}.$$

For a prime number $p \nmid N$ and $f \in S_k(\Gamma)$, we put

$$T_k(p)f = \lambda(p)f, \quad T_k(p^2)f = \lambda(p^2)f.$$

We define

$$H_p(t,f) := t^4 - \lambda(p)t^3 + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})t^2 - \lambda(p)p^{2k-3}t + p^{4k-6}.$$

Reference

T.Sugano, "On holomorphic cusp forms on quaternion unitary groups of degree 2", J.Fac.Sci.Univ.Tokyo 31.

Hecke eigenvalues of $T(5)$ and $T(5^2)$ on $S_k(U'(6))$ are as follows:

	$T(5)$	$T(5^2)$
wt 4	156	16561
wt 5	540	277225
	1140	835225
wt 6	4620	13785025
	2220	6369025
wt 7	13380	172290025
	7020	161796025
wt 8	36300	4018080625
	$114108 + 384\sqrt{1969}$	$8716867153 + 51634944\sqrt{1969}$
	$114108 - 384\sqrt{1969}$	$8716867153 - 51634944\sqrt{1969}$
wt 9	559260	203206513225
	353940	111952039225
	749460	362968807225
	-33540	10314697225

weight 4 $\quad S_4(U'(6)) = \mathbb{C}f, \quad f = -\frac{13}{288}(E_4 - {E_2}^2)$

We obtain

$$T_4(5)f = 156f, \qquad T_4(5^2)f = 16561f$$

$$H_5(t, f) = t^4 - 156t^3 + 7150t^2 - 487500t + 9765625.$$

weight 4 $\quad S_4(U'(6)) = \mathbb{C}f, \quad f = -\frac{13}{288}(E_4 - {E_2}^2)$

We obtain

$$T_4(5)f = 156f, \qquad T_4(5^2)f = 16561f$$

$$H_5(t,f) = (t-5^2)(t-5^3)(t^2-6t+5^5).$$

weight 4 $\quad S_4(U'(6)) = \mathbb{C}f, \quad f = -\frac{13}{288}(E_4 - {E_2}^2)$

We obtain

$$T_4(5)f = 156f, \qquad T_4(5^2)f = 16561f$$

$$H_5(t, f) = (t - 5^2)(t - 5^3)(t^2 - 6t + 5^5).$$

The spaces of 1-var. cusp forms of weight 6 are

$\dim S_6^{new}(\Gamma_0^{(1)}(2)) = 0,$

$\dim S_6^{new}(\Gamma_0^{(1)}(3)) = 1$

$g_1 = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6)$ *Sign* : $+$

$\dim S_6^{new}(\Gamma_0^{(1)}(6)) = 1$

$g_2 = q + 4q^2 - 9q^3 + 16q^4 - 66q^5 + O(q^6)$ *Sign* : $+$

weight 5 $\quad S_5(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_5(5)f_1 = 540f_1, \qquad T_5(25)f_1 = 277225f_1$$
$$T_5(5)f_2 = 1140f_2, \qquad T_5(25)f_2 = 835225f_2.$$

$$H_5(t, f_1) = t^4 - 540t^3 - 1250t^2 - 540 \cdot 5^7 \cdot t + 5^{14}$$
$$H_5(t, f_2) = t^4 - 1140t^3 + 448750t^2 - 1140 \cdot 5^7 \cdot t + 5^{14}.$$

weight 5 $\quad S_5(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_5(5)f_1 = 540f_1, \qquad T_5(25)f_1 = 277225f_1$$
$$T_5(5)f_2 = 1140f_2, \qquad T_5(25)f_2 = 835225f_2.$$

$$H_5(t, f_1) = (t - 5^3)(t - 5^4)(t^2 + 210t + 5^7)$$
$$H_5(t, f_2) = (t - 5^3)(t - 5^4)(t^2 - 390t + 5^7).$$

weight 5 $\quad S_5(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_5(5)f_1 = 540f_1, \quad T_5(25)f_1 = 277225f_1$$
$$T_5(5)f_2 = 1140f_2, \quad T_5(25)f_2 = 835225f_2.$$

$$H_5(t, f_1) = (t-5^3)(t-5^4)(t^2+210t+5^7)$$
$$H_5(t, f_2) = (t-5^3)(t-5^4)(t^2-390t+5^7).$$

The spaces of 1-var. cusp forms of weight 8 are

$\dim S_8^{new}(\Gamma_0^{(1)}(2)) = 1,$

$g_1 = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + O(q^6)$ $\quad$ *Sign* : $+$

$\dim S_8^{new}(\Gamma_0^{(1)}(3)) = 1$

$g_2 = q + 6q^2 - 27q^3 - 92q^4 + 390q^5 + O(q^6)$ $\quad$ *Sign* : $+$

$\dim S_8^{new}(\Gamma_0^{(1)}(6)) = 1$

$g_3 = q + 8q^2 + 27q^3 + 64q^4 - 114q^5 + O(q^6)$ $\quad$ *Sign* : $+$

weight 6 $\quad S_6(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_6(5)f_1 = 4620f_1, \qquad T_6(5^2)f_1 = 13785025f_1$$
$$T_6(5)f_2 = 2220f_2, \qquad T_6(5^2)f_2 = 6369025f_2$$

$$H_5(t, f_1) = t^4 - 4620t^3 + 7168750t^2 - 4620 \cdot 5^9 t + 5^{18}$$
$$H_5(t, f_2) = t^4 - 2220t^3 - 1831250t^2 - 2220 \cdot 5^9 t + 5^{18}$$

weight 6 $\quad S_6(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_6(5)f_1 = 4620f_1, \qquad T_6(5^2)f_1 = 13785025f_1$$
$$T_6(5)f_2 = 2220f_2, \qquad T_6(5^2)f_2 = 6369025f_2$$

$$H_5(t, f_1) = (t - 5^4)(t - 5^5)(t^2 - 870t + 5^9)$$
$$H_5(t, f_2) = (t - 5^4)(t - 5^5)(t^2 + 1530t + 5^9)$$

weight 6 $\quad S_6(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

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$$H_5(t, f_1) = (t - 5^4)(t - 5^5)(t^2 - 870t + 5^9)$$
$$H_5(t, f_2) = (t - 5^4)(t - 5^5)(t^2 + 1530t + 5^9)$$

The spaces of 1-var. cusp forms of weight 10 are

$$\mathrm{dim} S_{10}^{new}(\Gamma_0^{(1)}(2)) = 1,$$
$$g_1 = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + O(q^6) \qquad Sign : +$$
$$\mathrm{dim} S_{10}^{new}(\Gamma_0^{(1)}(3)) = 2$$
$$g_2 = q + 18q^2 + 81q^3 - 188q^4 - 1530q^5 + O(q^6) \qquad Sign : +$$
$$g_3 = q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + O(q^6) \qquad Sign : -$$
$$\mathrm{dim} S_{10}^{new}(\Gamma_0^{(1)}(6)) = 1$$
$$g_4 = q - 16q^2 + 81q^3 + 256q^4 + 2694q^5 + O(q^6) \qquad Sign : +$$

weight 7 $\quad S_7(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_7(5)f_1 = 13380 f_1, \quad T_7(25)f_1 = 172290025 f_1$$
$$T_7(5)f_2 = 7020 f_2, \quad T_7(25)f_2 = 161796025 f_2$$

$$H_5(t, f_1) = t^4 - 13380t^3 - 3031250t^2 - 13380 \cdot 5^{11} t + 5^{22}$$
$$H_5(t, f_2) = t^4 - 7020t^3 - 122281250t^2 - 7020 \cdot 5^{11} t + 5^{22}.$$

weight 7 $\quad S_7(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_7(5)f_1 = 13380f_1, \quad T_7(25)f_1 = 172290025f_1$$
$$T_7(5)f_2 = 7020f_2, \quad T_7(25)f_2 = 161796025f_2$$

$$H_5(t, f_1) = (t - 5^5)(t - 5^6)(t^2 + 5370t + 5^{11})$$
$$H_5(t, f_2) = (t - 5^5)(t - 5^6)(t^2 + 11730t + 5^{11}).$$

weight 7 $\quad S_7(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We obtain

$$T_7(5)f_1 = 13380 f_1, \quad T_7(25)f_1 = 172290025 f_1$$
$$T_7(5)f_2 = 7020 f_2, \quad T_7(25)f_2 = 161796025 f_2$$

$$H_5(t, f_1) = (t - 5^5)(t - 5^6)(t^2 + 5370t + 5^{11})$$
$$H_5(t, f_2) = (t - 5^5)(t - 5^6)(t^2 + 11730t + 5^{11}).$$

The spaces of 1-var. cusp forms of weight 12 are

$$\dim S_{12}^{new}(\Gamma_0^{(1)}(2)) = 0$$
$$\dim S_{12}^{new}(\Gamma_0^{(1)}(3)) = 1,$$
$$g_1 = q + 78q^2 - 243q^3 + 4036q^4 - 5370q^5 + O(q^6) \quad Sign : +$$
$$\dim S_{12}^{new}(\Gamma_0^{(1)}(6)) = 3,$$
$$g_2 = q + 32q^2 + 243q^3 + 1024q^4 + 3630q^5 + O(q^6) \quad Sign : +$$
$$g_3 = q - 32q^2 - 243q^3 + 1024q^4 + 5766q^5 + O(q^6) \quad Sign : +$$
$$g_4 = q - 32q^2 + 243q^3 + 1024q^4 - 11730q^5 + O(q^6) \quad Sign : -$$

weight 8 $\quad S_8(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3$

We obtain

$$\begin{aligned} T_8(5)f_1 &= 36300 f_1, & T_8(5^2)f_1 &= 4018080625 f_1 \\ T_8(5)f_2 &= \alpha^+ f_2, & T_8(5^2)f_2 &= \beta^+ f_2 \\ T_8(5)f_3 &= \alpha^- f_3, & T_8(5^2)f_3 &= \beta^- f_3 \end{aligned}$$

$$\left(\begin{aligned} \alpha^+, \alpha^- &= 114108 \pm 384\sqrt{1969}, \\ \beta^+, \beta^- &= 8716867153 \pm 51634944\sqrt{1969} \end{aligned} \right)$$

$$\begin{aligned} H_5(t, f_1) &= t^4 - 36300t^3 - 2944531250t^2 - 36300 \cdot 5^{13} t + 5^{26} \\ H_5(t, f_2) &= t^4 - \alpha^+ t^3 + ({\alpha^+}^2 - \beta^+ - 5^{12})t^2 - \alpha^+ 5^{13} t + 5^{26} \\ H_5(t, f_3) &= t^4 - \alpha^- t^3 + ({\alpha^-}^2 - \beta^- - 5^{12})t^2 - \alpha^- 5^{13} t + 5^{26} \end{aligned}$$

weight 8 $\quad S_8(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3$

We obtain

$$
\begin{aligned}
T_8(5)f_1 &= 36300 f_1, & T_8(5^2)f_1 &= 4018080625 f_1 \\
T_8(5)f_2 &= \alpha^+ f_2, & T_8(5^2)f_2 &= \beta^+ f_2 \\
T_8(5)f_3 &= \alpha^- f_3, & T_8(5^2)f_3 &= \beta^- f_3
\end{aligned}
$$

$$
\left(\begin{aligned} \alpha^+, \alpha^- &= 114108 \pm 384\sqrt{1969}, \\ \beta^+, \beta^- &= 8716867153 \pm 51634944\sqrt{1969} \end{aligned} \right)
$$

$$
\begin{aligned}
H_5(t, f_1) &= (t - 5^6)(t - 5^7)(t^2 + 57450t + 5^{13}) \\
H_5(t, f_2) &= (t - 5^6)(t - 5^7)(t^2 - (20358 + 384\sqrt{1969})t + 5^{13}) \\
H_5(t, f_3) &= (t - 5^6)(t - 5^7)(t^2 - (20358 - 384\sqrt{1969})t + 5^{13})
\end{aligned}
$$

weight 8 $\quad S_8(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3$

$$H_5(t, f_1) = (t-5^6)(t-5^7)(t^2+57450t+5^{13})$$
$$H_5(t, f_2) = (t-5^6)(t-5^7)(t^2-(20358+384\sqrt{1969})t+5^{13})$$
$$H_5(t, f_3) = (t-5^6)(t-5^7)(t^2-(20358-384\sqrt{1969})t+5^{13})$$

weight 8 $\quad S_8(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3$

$$H_5(t, f_1) = (t-5^6)(t-5^7)(t^2 + 57450t + 5^{13})$$
$$H_5(t, f_2) = (t-5^6)(t-5^7)(t^2 - (20358 + 384\sqrt{1969})t + 5^{13})$$
$$H_5(t, f_3) = (t-5^6)(t-5^7)(t^2 - (20358 - 384\sqrt{1969})t + 5^{13})$$

The spaces of 1-var. cusp forms of weight 14 are

$\dim S_{14}^{new}(\Gamma_0^{(1)}(2)) = 2$,

$g_1 = q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + O(q^6)$: $Sign-$

$g_2 = q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + O(q^6)$: $Sign+$

$\dim S_{14}^{new}(\Gamma_0^{(1)}(3)) = 3$

$g_3 = q - 12q^2 - 729q^3 - 8048q^3 - 30210q^5 + O(q^6)$: $Sign-$

$g_4 = q + \cdots + (20358 + 384\sqrt{1969})q^5 + O(q^6)$: $Sign+$

$g_5 = q + \cdots + (20358 - 384\sqrt{1969})q^5 + O(q^6)$: $Sign+$

$\dim S_{14}^{new}(\Gamma_0^{(1)}(6)) = 1$

$g_6 = q + 64q^2 - 729q^3 + 4096q^4 + 54654q^5 + O(q^6)$: $Sign+$

All the results agreed perfectly with our lifting conjecture.

• • •

Next, we want numerical evidence of Conjecture 2.

We can find a non-lift eigenform in $S_9(U'(6))$.
We will obtain an example of coincidence of Euler 5-factor.

weight 9: We have Hecke eigenforms f_1, f_2, f_3, f_4 such that

$$S_9(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4$$

and

$$\begin{aligned} T_9(5)f_1 &= 559260 f_1, & T_9(25)f_1 &= 203206513225 f_1, \\ T_9(5)f_2 &= 749460 f_2, & T_9(25)f_2 &= 362968807225 f_2, \\ T_9(5)f_3 &= 353940 f_3, & T_9(25)f_3 &= 111952039225 f_3, \\ T_9(5)f_4 &= -33540 f_4, & T_9(25)f_4 &= 10314697225 f_4. \end{aligned}$$

Hence we have

$$H_5(t, f_1) = t^4 - 559260 t^3 + 103461718750 t^2 - 559260 \cdot 5^{15} t + 5^{30}$$

$$H_5(t, f_2) = t^4 - 749460 t^3 + 192617968750 t^2 - 749460 \cdot 5^{15} t + 5^{30}$$

$$H_5(t, f_3) = t^4 - 353940 t^3 + 7217968750 t^2 - 353940 \cdot 5^{15} t + 5^{30}$$

$$H_5(t, f_4) = t^4 + 33540 t^3 - 15293281250 t^2 + 33540 \cdot 5^{15} t + 5^{30}.$$

weight 9: We have Hecke eigenforms f_1, f_2, f_3, f_4 such that

$$S_9(U'(6)) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4$$

and

$$\begin{aligned}
T_9(5)f_1 &= 559260 f_1, & T_9(25)f_1 &= 203206513225 f_1,\\
T_9(5)f_2 &= 749460 f_2, & T_9(25)f_2 &= 362968807225 f_2,\\
T_9(5)f_3 &= 353940 f_3, & T_9(25)f_3 &= 111952039225 f_3,\\
T_9(5)f_4 &= -33540 f_4, & T_9(25)f_4 &= 10314697225 f_4.
\end{aligned}$$

Hence we have

$$H_5(t, f_1) = (t - 5^7)(t - 5^8)(t^2 - 90510t + 5^{15})$$

$$H_5(t, f_2) = (t - 5^7)(t - 5^8)(t^2 - 280710t + 5^{15})$$

$$H_5(t, f_3) = (t - 5^7)(t - 5^8)(t^2 + 114810t + 5^{15})$$

$$H_5(t, f_4) = t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15} t + 5^{30}.$$

$H_5(t, f_4)$ doesn't have such factorization.

We obtained

$$H_5(t, f_4) = t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30}.$$

On the other hand,

By Theorem 1, we have $\dim S_9(K(6)) = 3$. Gritsenko's lift span the two-dimensional subspace.

g_1, g_2	:	two Gristenko lift eigenforms
g_3	:	non-lift eigenform

Poor and Yuen made calculations on $S_9(K(6))$ by using their technique, and kindly informed us of the data of Hecke eigenvalues of g_3.

$$H_5(t, g_3) = t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30}$$

- We calculated also for $S_{10}(U'(6))$.

$\boxed{S_{10}(U'(6))}$ $\quad \dim S_{10}(U'(6)) = 6$

$f_1, \cdots, f_5$: lift eigenforms
f_6 : non-lift eigenform

$$H_5(t, f_6) = t^5 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34}$$

- Poor and Yuen calculated also for $S_{10}(K(6))$.

$\boxed{S_{10}(K(6))}$ $\quad \dim S_{10}(K(6)) = 4$

g_1, g_2, g_3 : lift eigenforms
g_4 : non-lift eigenform

$$H_5(t, g_4) = t^5 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34}$$

- We calculated also for $S_{11}(U'(6))$.

$\boxed{S_{11}(U'(6))}$ $\dim S_{11}(U'(6)) = 6$

$f_1, \cdots, f_4$: lift eigenforms
f_5, f_6 : non-lift eigenforms

$$H_5(t, f_5) = t^4 + 222420t^3 + 21376386718750t^2 + 222420 \cdot 5^{19} t + 5^{38}$$
$$H_5(t, f_6) = t^4 - 5029620t^3 + 11396230468750t^2 - 5029620 \cdot 5^{19} + 5^{38}$$

- Poor and Yuen calculated also for $S_{11}(K(6))$.

$\boxed{S_{11}(K(6))}$ $\dim S_{11}(K(6)) = 5$

g_1, g_2, g_3 : lift eigenforms
g_4, g_5 : non-lift eigenforms

$$H_5(t, g_4) = t^4 + 222420t^3 + 21376386718750t^2 + 222420 \cdot 5^{19} t + 5^{38}$$
$$H_5(t, g_5) = t^4 - 5029620t^3 + 11396230468750t^2 - 5029620 \cdot 5^{19} + 5^{38}$$