Shifted Convolution of Cusp Forms with θ-Series

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1 Introduction and Main Results

♢ Shifted Convolution Sum

- Suppose that $\lambda_1(n)$ and $\lambda_2(n)$ are two (multiplicative) arithmetic functions, and $b \geq 0$ is an integer.
- It is a classical and important problem in analytic number theory to study the shifted convolution sum
$$\sum_{n \leq x} \lambda_1(n)\lambda_2(n+b).$$
- Difficulty: The shift parameter b destroys the multiplicativity.
- There are a large number of papers in this direction, which have many important applications.

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♢ Binary Additive Divisor Problem

- When $\lambda_1(n) = \lambda_2(n) = \tau(n)$, the Dirichlet divisor function, this problem is the so-called binary additive divisor problem $D(x; b)$.
  - Ingham [1927]:

$$D(x; b) \sim (1 + o(1))c\sigma_{-1}(b)x(\log x)^2.$$

  - Esterman [1930]: Improved this to an asymptotic expansion by observing a relation between $D(x; b)$ and the Kloosterman sum.
  - Atkinson [1941]: Found the importance of uniformity with respect to the shift parameter b and a relation between the error term in $D(x; b)$ and the power mean of $\zeta(1/2 + it)$.

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♢ Binary Additive Divisor Problem

- – Heath-Brown [1978]: Weil's bound led to a better uniform result on $D(x;b)$, and then an asymptotic formula for the fourth power mean of $\zeta(1/2+it)$ with error term $O(T^{\frac{7}{8}+\varepsilon})$.
  – Deshouillers and Iwaniec [1982]: The appearance of Kuznetsov's trace formula changed the situation dramatically by transforming sums of Kloosterman sums into bilinear forms of Fourier coefficients of cusp forms.
  – Further developments: Jutila [1993], Motohashi [1994], Duke, Friedlander, Iwaniec [1994], Ivić and Motohashi [1995], Meurman [2001]$\cdots\cdots$

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♢ Its Analog for Fourier Coefficients

- When $\lambda_1(n) = \lambda_2(n) = \lambda_f(n)$, Fourier coefficient of holomorphic or Maass cusp form f, it is an analog for Fourier coefficients of cusp forms of the additive divisor problem.
- Since Selberg's seminal paper [1965], this sum has been investigated extensively. See e.g. Good [1982], Jutila [1996, 1997], Sarnak [1994], Liu and Ye [2002], Harcos [2003], Lau, Liu and Ye [2006], Blomer and Harcos [2008], Holowinsky [2010], $\cdots\cdots$
- Non-trivial bound of this sum often has deep implications: e.g. subconvexity and equidistribution (QUE).

♢ Mixed Shifted Convolution Sum

- Let

$$r_\ell(n) := \left|\left\{(n_1,\ldots,n_\ell)\in\mathbb{Z}^\ell : n_1^2+\cdots+n_\ell^2 = n\right\}\right|.$$

- For each holomorphic cusp form f of weight k and level N, we write its Fourier expansion at ∞:

$$f(z) = \sum_{n\ge 1}\lambda_f(n)n^{(k-1)/2}\mathrm{e}(nz),$$

- Recently Luo considered the shifted convolution sum of cusp forms with theta series

$$\sum_{n\le x}\lambda_f(n+b)r_\ell(n) := \mathscr{S}_{f,b,\ell}(x).$$

- Luo [2011]: For $\ell \geq 2$, $k \geq \ell/2+3$ and $\varepsilon > 0$, we have

$$\mathscr{S}_{f,b,\ell}(x) \ll x^{\ell/2-\vartheta_\ell+\varepsilon}, \tag{1}$$

where $\vartheta_\ell := (\ell-1)/(4g+4)$ and g is the smallest integer such that $g \geq (\ell+1)/2$ and the implied constant depends on f, b, ℓ and ε.

- In particular

$$\vartheta_2 = \frac{1}{12}, \quad \vartheta_3 = \frac{1}{6}, \quad \vartheta_4 = \frac{3}{16},$$

$$\vartheta_5 = \frac{1}{4}, \quad \vartheta_6 = \frac{1}{4}, \quad \vartheta_\ell = \frac{\ell-1}{2\ell+6} < \frac{1}{2} (\ell \geq 7).$$

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- Luo's idea is try to generalize the classical Voronoi formula for $r_\ell(n)$, and then combined this formula with upper bound for the Salié sum to derive result (1).
- Based on a series of work on "the divisor problems related to the Epstein zeta-function": Guangshi Lü, Jie Wu and Wenguang Zhai [Bull. London Math. Soc. 2010; JNT 2011; Acta Arith. 2012; Quart. J. Math. 2012], we try to explore the regularity of $r_\ell(n)$ by the circle method in analytic number theory and Siegel's mass formula.

- Thanking to these classic tools of analytic number theory, we can show that the influence of $r_\ell(n)$ to the bound

$$\sum_{n\le x}\lambda_f(n)\ll_f x^{1/3}(\log x)^{2/(\sqrt{\pi}\Gamma(5/2))-1}$$

is rather little.

- G.S. Lü, J. Wu and W.G. Zhai [2013]

Theorem 1. Let f be a cusp form of weight k and level N and let $\ell \geq 2$ be an integer. For any $\varepsilon > 0$, we have

$$\mathscr{S}_{f,b,\ell}(x) \ll_{f,\ell,\varepsilon} x^{\ell/2-\vartheta_\ell+\varepsilon} \tag{2}$$

uniformly for $x \geq 2$ and $0 \leq b \leq x$, where

$$\vartheta_3 = \frac{1}{4}, \quad \vartheta_4 = \frac{1}{2}, \quad \vartheta_5 = \frac{1}{2}, \quad \vartheta_\ell = \frac{2}{3}(\ell \geq 6).$$

In addition, if we assume $N = 1$, then (2) holds for $\ell = 2$ with $\vartheta_2 = \frac{1}{6}$.

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For comparison, we have

Our Result	Luo's Result
• $\vartheta_2 = \frac{1}{6}, (N = 1)$	• $\vartheta_2 = \frac{1}{12}$
• $\vartheta_3 = \frac{1}{4}$	• $\vartheta_3 = \frac{1}{6}$
• $\vartheta_4 = \frac{1}{2}$	• $\vartheta_4 = \frac{3}{16}$
• $\vartheta_5 = \frac{1}{2}$	• $\vartheta_5 = \frac{1}{4}$
• $\vartheta_6 = \frac{2}{3}$	• $\vartheta_6 = \frac{1}{4}$
• $\vartheta_\ell = \frac{2}{3}$ $\ell \geq 7$	• $\vartheta_\ell < \frac{1}{2}$ $\ell \geq 7$

Our result improves Luo's result above in three directions:

- Enlarge the exponent ϑ_ℓ;
- Relax the restricted condition $k \geq \ell/2 + 3$;
- Remove the dependence of b.
- **More general. $r_\ell(n)$-aspect:** Let $Q(\mathbf{y})$ be a positive definite quadratic form $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y}$. For each $n \geq 1$, define

$$r(n, Q) := \left|\left\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\right\}\right|.$$

Similar to $\mathscr{S}_{f,b,\ell}(x)$, we define

$$\mathscr{S}_{f,b,Q}(x) := \sum_{n \leq x} \lambda_f(n+b) r(n, Q).$$

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- **More general. f-aspect:** Define $\mathcal{F}$ to be a class of cusp forms $f(z)$, which consists of holomorphic cusp forms with respect to any finite volume discrete subgroup (such that ∞ is a singular cusp of width 1), any positive real weight and any multiplier systems, as well as Maass cusp forms of any weight and any level.
- Then our result implies for any $f \in \mathcal{F}$ and any general $Q(\mathbf{y})$ that

$$\mathscr{S}_{f,b,Q}(x) \ll_{f,Q,\varepsilon} x^{\ell/2-1/2+\varepsilon}$$

holds uniformly for $1 \leq b \leq x$, provided $\ell \geq 5$.

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- A natural question is what should be the best bound for $\mathscr{S}_{f,b,\ell}(x)$.
- **Conjecture.** Let f be a cusp form of weight k and level N. Let $b \geq 0$ and $\ell \geq 3$ be two integers. For any $\varepsilon > 0$, we have

$$\mathscr{S}_{f,b,\ell}(x) \ll_{f,b,\ell,\varepsilon} x^{\ell/2-3/4+\varepsilon}$$

for $x \to \infty$.

- It seems rather difficult to establish this conjecture. However we can prove that the conjectured bound is true on average for $\ell \geq 5$.

Theorem 2. If $\ell \geq 6$, then we have

$$\int_1^X |\mathscr{S}_{f,b,Q}(x)|^2 \mathrm{d}x = \frac{C_{f,b,Q}}{\ell - 1/2} X^{\ell-1/2} + O_{f,Q}\big(bX^{\ell-3/2} + X^{\ell-7/12}(\log X)^{1/2}\big).$$

Furthermore, we have

$$\int_1^X |\mathscr{S}_{f,b,5}(x)|^2 \mathrm{d}x \ll_{f,\varepsilon} X^{\ell-1/2+\varepsilon}.$$

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2 Proof of Theorem 1: Case $\ell \geq 3$

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- **Aim: By a rather simple proof, to show that $\vartheta_3 = \frac{1}{4}$ and $\vartheta_\ell = \frac{1}{2}$ for $\ell \geq 4$.**
- Let
$$F(\alpha) := \sum_{n \leq x} \lambda_f(n+b)\mathrm{e}(-\alpha n)$$
and
$$S(\alpha) := \sum_{|m| \leq x^{1/2}} \mathrm{e}(\alpha m^2).$$
- Then it is easy to see
$$\mathscr{S}_{f,b,\ell}(x) = \int_0^1 F(\alpha)S(\alpha)^\ell \mathrm{d}\alpha.$$

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- **Square Root Cancellation:** Let f be a cusp form of weight k and level N. The estimate

$$\sum_{n\le x}\lambda_f(n)\mathrm{e}(\alpha n)\ll_f x^{1/2}\log x$$

holds uniformly for $\alpha\in\mathbb{R}$.

- In addition, it is not hard to show

$$\int_0^1|S(\alpha)|^2\mathrm{d}\alpha=\sum_{\substack{|m|\le x^{1/2}\\ m^2=n^2}}\sum_{|n|\le x^{1/2}}1\ll x^{1/2},$$

$$\int_0^1|S(\alpha)|^{2d}\mathrm{d}\alpha\le\sum_{n\le dx}r_d(n)^2\ll x^{d-1}.$$

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- Thus we can show

$$\begin{aligned}
&\mathscr{S}_{f,b,3}(x)\\
&\ll x^{1/2}(\log x)\Big(\int_0^1 |S(\alpha)|^2 \mathrm{d}\alpha \int_0^1 |S(\alpha)|^4 \mathrm{d}\alpha\Big)^{\frac{1}{2}}\\
&\ll x^{3/2-1/4}\log x.
\end{aligned}$$

- For $\ell \geq 4$

$$\begin{aligned}
&\mathscr{S}_{f,b,\ell}(x)\\
&\ll x^{\frac{1}{2}}(\log x)\Big(\int_0^1 |S(\alpha)|^4 \mathrm{d}\alpha \int_0^1 |S(\alpha)|^{2(\ell-2)} \mathrm{d}\alpha\Big)^{\frac{1}{2}}\\
&\ll x^{\ell/2-1/2}\log x,
\end{aligned}$$

namely we can take $\vartheta_3 = \frac{1}{4}$ and $\vartheta_\ell = \frac{1}{2}$ for $\ell \geq 4$.

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3 Proof of Theorem 1: Case $\ell \geq 6$

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Aim: When $\ell \geq 6$, further improve the exponent $\vartheta_\ell = \frac{1}{2}$.

Lemma 1. (Approximate Voronoi Formula) Let f be a cusp form of weight k and level N, $(h, q) = 1$

$$A(x, h/q) := \sum_{n \leq x}{}' \lambda_f(n) \mathrm{e}_q(hn).$$

Then for any $\varepsilon > 0$ we have

$$\begin{aligned} &A(x, h/q) \\ &= \frac{q^{1/2} x^{1/4}}{\sqrt{2}\pi} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \mathrm{e}_q(-\overline{h}n) \cos\left(\frac{4\pi\sqrt{nx}}{q} - \frac{\pi}{4}\right) \\ &+ O_{f,\varepsilon}\left(\frac{q x^{1/2+\varepsilon}}{M^{1/2}}\right) \end{aligned}$$

uniformly for $1 \leq q \leq x$ and $1 \leq M \ll x$.

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Lemma 2. Let $\ell \geq 2$, $\mathbf{y} := (y_1, \ldots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod 2$ for $1 \leq i \leq \ell$. The positive definite quadratic form $Q(\mathbf{y})$ is defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y}$. For each $n \geq 1$, define

$$r(n, Q) := \left|\left\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\right\}\right|.$$

Then for $\ell \geq 4$ we have

$$r(n, Q) = \sigma_Q n^{\frac{\ell}{2}-1} \sum_{q=1}^{\infty} \sum_{h=1}^{q}{}^{*} S\left(\frac{hQ}{q}\right) \frac{\mathrm{e}(-\frac{hn}{q})}{q^\ell} + O\left(n^{\frac{\ell}{4}-\delta_\ell+\varepsilon}\right),$$

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where

$$S(Q) := \sum_{0 \le y_1, \ldots, y_\ell \le q-1} \mathrm{e}(Q(\mathbf{y})),$$

$$\sigma_Q := \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2)\sqrt{|\mathbf{A}|}},$$

$$\delta_\ell := \begin{cases} \frac{1}{4} & \text{if } \ell \text{ is odd,} \\ \frac{1}{2} & \text{if } \ell \text{ is even,} \end{cases}$$

and $\sum^*$ means the sum is over $1 \le h \le q$ with $(h, q) = 1$. Furthermore we have

$$S(hQ/q) \ll q^{\ell/2} \quad ((h, q) = 1).$$

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Proof of Case $\ell \geq 6$. One can easily deduce that

$$
\begin{aligned}
&\mathscr{S}_{f,b,\ell}(x) \qquad (+\textbf{Lemma 2}) \\
\ll\ & \sum_{q\geq 1} \frac{1}{q^{\frac{\ell}{2}}} \sum_{\substack{1\leq h\leq q \\ (h,q)=1}} \Big| \sum_{n=1+b}^{x+b} n^{\frac{\ell}{2}-1} \lambda_f(n) e_q(-hn) \Big| \\
& + x^{\ell/4-\delta_\ell+1+\varepsilon} \qquad (+\textbf{Lemma 1}) \\
\ll_{f,\varepsilon}\ & x^{\ell/2-2/3+\varepsilon} \sum_{q\geq 1} \frac{1}{q^{\ell/2-5/3}} + x^{\ell/4-\delta_\ell+1+\varepsilon} \\
\ll_{f,\varepsilon}\ & x^{\ell/2-2/3+\varepsilon} \qquad (\textbf{recall } \ell \geq 6).
\end{aligned}
$$

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4 Proof of Theorem 1: Case $\ell = 2$

In order to deal with case $\ell = 2$, we need the following result.

Lemma 3. Let f be a cusp form of weight k and level 1. Then the estimate

$$S_f(x; a, q) := \sum_{\substack{n \le x \\ n \equiv a \pmod{q}}} \lambda_f(n) \ll_f x^{1/3+\varepsilon}$$

holds uniformly for $x \ge 1$ and $q \ge a \ge 1$.

Remark. Note that previous similar results proved by R.A. Smith needs a restricted condition $(a, q) = 1$.

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Here we are able to relax this restricted condition $(a, q) = 1$ by proving an auxiliary lemma.

Lemma 4. Let $m \geq 2$ be a positive integer. There is an arithmetic function $h_m(n)$ such that

$$\begin{aligned} h_m(n) &= 0 \quad \text{if } \exists\, p \text{ such that } p \mid n \text{ and } p \nmid m, \\ |h_m(n)| &\leq \tau(m)\tau_4(n) \quad \text{if } n \mid m^\infty, \\ \lambda_f(mn) &= \sum_{d|n} h_m(d)\lambda_f(n/d), \end{aligned}$$

where $\tau_k(n)$ denotes the number of solutions of $n = n_1 \cdots n_k$ with positive numbers $n_1, \ldots, n_k$, and $\tau(n) := \tau_2(n)$.

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Proof of Case $\ell = 2$.

By the classical expression

$$r_2(n) = 4\sum_{d|n}\chi(d)$$

where $\chi(n)$ is the non trivial Dirichlet character modulo 4, we can write

$$\mathscr{S}_{f,b,2}(x) = 4S_1 + 4S_2 - 4S_3,$$

where

$$S_1 := \sum_{d\le\sqrt{x}} \sum_{dm\le x} \chi(d)\lambda_f(dm+b),$$

$$S_2 := \sum_{m\le\sqrt{x}} \sum_{dm\le x} \chi(d)\lambda_f(dm+b),$$

$$S_3 := \sum_{d\le\sqrt{x}} \sum_{m\le\sqrt{x}} \chi(d)\lambda_f(dm+b).$$

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By Lemma 3, we have (uniformly for $0 \le b \le x$)

$$S_1 = \sum_{d \le \sqrt{x}} \chi(d) \sum_{\substack{n \le x+b \\ n \equiv b (\mathrm{mod}\, d)}} \lambda_f(n) \ll \sum_{d \le \sqrt{x}} (x+b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}.$$

$$S_3 = \sum_{d \le \sqrt{x}} \chi(d) \sum_{\substack{n \le d\sqrt{x}+b \\ n \equiv b (\mathrm{mod}\, d)}} \lambda_f(n) \ll \sum_{d \le \sqrt{x}} (d\sqrt{x}+b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}.$$

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Note that

$$\chi(d) = 1 \quad \text{if } d \equiv 1 \pmod 4,$$
$$\chi(d) = -1 \quad \text{if } d \equiv 3 \pmod 4,$$
$$\chi(d) = 0 \quad \text{if } 2 \mid d,$$

$$S_2 = \sum_{m \le \sqrt{x}} \sum_{(4d+1)m \le x} \lambda_f((4d+1)m+b)$$
$$- \sum_{m \le \sqrt{x}} \sum_{(4d+3)m \le x} \lambda_f((4d+3)m+b)\Big).$$

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By Lemma 3, we have (uniformly for $0 \le b \le x$)

$$S_2 = \sum_{m \le \sqrt{x}} \sum_{\substack{n \le x+b \\ n \equiv m+b \pmod{4m}}} \lambda_f(n) - \sum_{m \le \sqrt{x}} \sum_{\substack{n \le x+b \\ n \equiv 3m+b \pmod{4m}}} \lambda_f(n) \ll \sum_{m \le \sqrt{x}} (x+b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}.$$

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5 About the Proof of Theorem 2

- It suffices for us to evaluate $\int_{\sqrt{X}}^{X} |\mathscr{S}_{f,b,Q}(x)|^2 \mathrm{d}x$.
- By Lemma 2 (not exactly), we have

$$\mathscr{S}_{f,b,Q}(x) = \sigma_Q \sum_{q=1}^{\infty} \sum_{h=1}^{q}{}^{*} S\left(\frac{hQ}{q}\right) \frac{\mathrm{e}_q(bh)}{q^\ell} A_{\ell,b}(x,-h/q) + O\big(\delta_Q x^{\ell/4-\delta_\ell+1+\varepsilon}\big).$$

Here

$$A_{\ell,b}(x,-\frac{h}{q}) := \sum_{1+b\le n\le x+b} (n-b)^{\frac{\ell}{2}-1} \lambda_f(n) \mathrm{e}_q(-hn).$$

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- When $q > X^{\frac{1}{2}}$, the contribution of

$$\sum_{q>X^{1/2}} \frac{1}{q^\ell} \sum_{h=1}^{q}{}^{*} \left| S\left(\frac{hQ}{q}\right) A_{\ell,b}(x,-h/q) \right|$$

to $\mathscr{S}_{f,b,Q}(x)$ is negligible.

- Then after some arguments, we have a relatively simple formula

$$\mathscr{S}_{f,b,Q}(x) = \sigma_Q x^{\frac{\ell}{2}-1} \sum_{q\le X^{1/2}} \sum_{h=1}^{q}{}^{*} S\Big(\frac{hQ}{q}\Big) \frac{\mathrm{e}_q(bh)}{q^\ell} \times \\ \times A(x+b,-h/q) + O(R_\ell(X)).$$

Here $A(x,h/q) := \sum'_{n\le x} \lambda_f(n)\mathrm{e}_q(hn)$.

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- Now we split the sum over $q \le X^{1/2}$ into two parts according to $q \le X^{1/6}$ or $X^{1/6} < q \le X^{1/2}$.

$$\mathscr{S}_{f,b,Q}(x) = S_1(x) + S_2(x) + O(R_\ell(X)).$$

- Evaluate the integral $\int_{\sqrt{X}}^{X} |S_1(x)|^2 \mathrm{d}x$.

After opening up the mean square, evaluating the diagonal terms, and estimating non-diagonal terms, eventually we have

$$\int_{\sqrt{X}}^{X} |S_1(x)|^2 \mathrm{d}x = C_{f,b,Q} \int_1^X x^{\ell-2}(x+b)^{1/2} \mathrm{d}x \\ +O\big(R_\ell^*(X)\big).$$

- Estimate the integral $\int_{\sqrt{X}}^{X}|S_2(x)|^2\mathrm{d}x$.
- We need one mean value result for $A(x,\frac{h}{q})$ due to Jutila

$$\int_1^X |A(x,\frac{h}{q})|^2\mathrm{d}x = \frac{1}{(4k+2)\pi^2}\sum_{n=1}^{\infty}\frac{|\lambda_f(n)|^2}{n^{\frac{3}{2}}}qX^{\frac{3}{2}} + O_{f,\varepsilon}(q^{3/2}X^{\frac{5}{4}+\varepsilon}+q^2X^{1+\varepsilon}).$$

- This determines that

$$\int_{\sqrt{X}}^{X}|S_2(x)|^2\mathrm{d}x \ll \begin{cases} X^{\ell-2/3}\mathcal{L} & \text{if } \ell\geq 6,\\ X^{\ell-1/2+\varepsilon} & \text{if } \ell = 5.\end{cases}$$

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Thank You!