Derivative of Siegel modular forms and Jacobi forms by connections

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1. Connections and application to classical modular forms

We will explain by connections the two well-known derivative operators

$$D_1 f := \frac{df}{dz} - \frac{\sqrt{-1}k}{y} f \quad \text{and} \quad D_2 f := \frac{df}{dz} - \sqrt{-1}kG_2(z)f$$

for a modular form f of weight $2k$, where the first is non-holomorphic modular form of weight $2k+2$ and the second is holomorphic modular form of weight $2k+2$. First we recall the definition of a connection.

1.1. Connections in differential geometry

Suppose E is a q-dimensional real vector bundle on a smooth manifold M, and $\Gamma(E)$ is the set of smooth sections of E on M. Let $T^*(M)$ be the cotangent space of M. A connection on the vector bundle E is a map

$$D: \quad \Gamma(E) \longrightarrow \Gamma(T^*(M) \otimes E),$$

which satisfies the following conditions

1. For any $s_1, s_2 \in \Gamma(E)$

$$D(s_1 + s_2) = D(s_1) + D(s_2).$$

2. For any $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$

$$D(\alpha s) = d\alpha \otimes \alpha D(s).$$

If M has a generalized Riemannian metric $G = \sum_{i,j} g_{ij} du^i du^j$, where $\{du^i\}$ is a series of coordinate on M, by the fundamental theorem of Riemannian geometry, M has a unique torsion-free (i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$) and metric-compatible connection D (i.e. D(G)=0), called Levi-Civita connection of M. The coefficients Γ^k_{ij} of the Levi-Civita connection are given by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right)$$

where g^{ij} are elements of the matrix $(g^{ij}) := (g_{ij})^{-1}$. We have

$$D(dz_k) = -\sum_{i,j} \Gamma^k_{ij} dz_i dz_j.$$

The following result is basic and useful in the application of connections to automorphic forms.

Let Γ be a group, (M, G) a Riemannian manifold and D the Levi-Civita connection on M. If Γ has a smooth left action on M such that $G(\sigma_\star X, \sigma_\star Y) = G(X, Y)$ for all $\sigma \in \Gamma, X, Y \in T(M)$, i.e. G is an Γ-invariant metric, then

$$\sigma D = D\sigma \qquad (\sigma \in \Gamma).$$

1.2. Levi-Civita connection on upper half plane

On the upper half plane $\mathbb{H}$, there is the $\mathrm{SL}_2(\mathbb{R})$-invariant metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz \cdot d\bar{z}}{y^2}, \quad \text{where} \quad z = x + iy,$$

and $\{dz, d\bar{z}\}$ is a series of coordinate on $\mathbb{H}$. Thus by fundamental theorem of Riemannian geometry, there is the Levi-Civita connection on $\mathbb{H}$.

Now we compute the connection coefficients Γ_{ij}^{k} of Levi-Civita connection by the formula above. In the upper half plane, we have

$$ds^2 = \frac{dz \cdot d\bar{z}}{y^2} = (dz, d\bar{z}) \begin{pmatrix} 0 & \frac{1}{2y^2} \\ \frac{1}{2y^2} & 0 \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}.$$

By the formula in last subsection, we can get (notice that $\frac{\partial z}{\partial \bar{z}} = 0, \frac{\partial \bar{z}}{\partial z} = 0$)

$$\Gamma^1_{1,1} = \frac{\sqrt{-1}}{y}, \quad \Gamma^1_{2,1} = \Gamma^1_{1,2} = 0, \quad \Gamma^1_{2,1} = \Gamma^1_{1,2} = 0,$$

and

$$\Gamma^1_{2,2} = \Gamma^2_{1,1} = 0, \quad \Gamma^2_{2,2} = -\frac{\sqrt{-1}}{y}.$$

So we have for the Levi-Civita connection D satisfying

$$D(dz) = -\frac{\sqrt{-1}}{y} dz \cdot dz, \quad D(d\bar{z}) = \frac{\sqrt{-1}}{y} d\bar{z} \cdot d\bar{z}$$

and for any holomorphic function f on $\mathbb{H}$,

$$D(f(dz)^k) = df(dz)^k + f \cdot D(dz)^k = \left(\frac{df}{dz} - \frac{\sqrt{-1}k}{y} f \right) (dz)^{k+1}.$$

If f is a modular form of weight $2k$, we have $f \cdot (dz)^k$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. Noticing that for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we have

$$\gamma D = D\gamma, \text{ and as } d(\gamma z) = \frac{dz}{(cz+d)^2},$$

we see that $D(f \cdot (dz)^k)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Then $\frac{df}{dz} - \frac{\sqrt{-1}k}{y} f$ is a non-holomorphic modular form of weight $2k+2$. We explain the first differential operator by the connection.

1.3. Modular connections

However, we can't get the second (holomorphic) operator in the way above. The main reason is that the conclusion $\gamma D = D\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$ is too strong. To study derivative of modular forms, we only need $\gamma D = D\gamma$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. So we introduce a weaker notion, modular connection, to get the second operator.

A connection D is determined by the connection coefficients. Put the connection coefficients together, we get the connection matrix ω, which determine the connection D completely. $\gamma D = D\gamma$ is equivalent to a transformation formula of ω for $\gamma \in \mathrm{SL}_2(\mathbb{R})$. We only require ω to satisfy the transformation formula for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Inversely if such ω is given, then we can get a connection D, which satisfies $\gamma D = D\gamma$ only for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Write $dz_1 = dz$ and $dz_2 = d\bar{z}$. The connection matrix ω of a connection D on $\mathbb{H}$ is determined by the equality

$$(D(dz_1), D(dz_2)) = -(dz_1, dz_2)\omega.$$

It is easy to see $\omega = (\omega_i^j)_{2\times 2}$, where $\omega_i^j = \Gamma_{i1}^j dz_1 + \Gamma_{i2}^j dz_2$. For $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, we define $S = S(\gamma)$ by the transformation

$$(d(\gamma z_1), d(\gamma z)) = (dz_1, dz_2)S.$$

It is easy to show that for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we have $\gamma D = D\gamma$ if and only if $\gamma(\omega) = -S^{-1}dS + S^{-1}\omega S$.

For the Levi-Civita connection D, since the matrices ω and S are diagonal matrices, the equality of matrices $\gamma(\omega) = -S^{-1}dS + S^{-1}\omega S$ is actually the following two equalities

$$\gamma(\omega_1^1) = -S_1^{-1}dS_1 + \omega_1^1 \quad \text{and} \quad \gamma(\omega_2^2) = -S_2^{-1}dS_2 + \omega_2^2,$$

where $\omega_i^i = \Gamma_{i,i}^i dz_i$, and $S_i = (cz_i + d)^{-2}$ for $i = 1, 2$. In fact, for any connection D, if it sends (r, s) form to (r, s) form, then D has the decomposition $D = D^{1,0} + D^{0,1}$, where $D^{1,0}$ is the holomorphic part. The two equalities above give $D^{1,0}$ and $D^{0,1}$ respectively.

Naturally we give the following definition: The modular connection coefficients on $\mathbb{H}$ is a $\mathbb{C}^{\infty}$-function Γ such that $\omega = \Gamma dz$ and

$$\gamma(\omega) = -S^{-1}dS + \omega, \quad \forall \gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The corresponding connection D is called a modular connection, which satisfies $\gamma D = D\gamma$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. If Γ is a holomorphic function, we also call the connection D to be holomorphic.

Notice that here we only consider the holomorphic part, and ignore the part on $d\bar{z}$. Let us see what condition the function Γ should satisfy.

Since $dS = -\frac{2c}{(cz+d)^3}dz$ and $\omega = \Gamma dz$, we have

$$\gamma(\omega) = -S^{-1}dS + \omega \Longleftrightarrow \frac{\gamma(\Gamma)}{(cz+d)^2} = \Gamma + \frac{2c}{cz+d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. But we have known $\Gamma = \frac{\sqrt{-1}}{y}$ and $\Gamma = \sqrt{-1}G_2(z)$ satisfy the equality. These Γ give us two modular connections D_1 and D_2.

We have

$$D_1(f dz) = (\frac{df}{dz} - \frac{\sqrt{-1}}{y} f)(dz)^2, \quad D_2(f dz) = (\frac{df}{dz} - \sqrt{-1} G_2(z) f)(dz)^2.$$

Furthermore

$$D_1(f(dz)^k) = (\frac{df}{dz} - \frac{\sqrt{-1}}{y} f)(dz)^{k+1},$$

$$D_2(f(dz)^k) = (\frac{df}{dz} - \sqrt{-1} G_2(z) f)(dz)^{k+1}.$$

Since $\gamma D_i = D_i \gamma, \forall \gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $f(dz)^k$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ if f is a modular form of weight $2k$, we get $\frac{df}{dz} - \frac{\sqrt{-1}}{y} f$ and $\frac{df}{dz} - \sqrt{-1} G_2(z) f$ are modular forms of weight $2k+2$ and the former is non-holomorphic and the latter is holomorphic.

In addition, $\sqrt{-1}G_2(z)$ is the unique holomorphic function Γ on $\mathbb{H}$ which satisfies

$$\frac{\gamma(\Gamma)}{(cz+d)^2} = \Gamma + \frac{2c}{cz+d}, \ \ \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

This is because $\gamma(\sqrt{-1}G_2(z) - \Gamma) = (cz+d)^2(\sqrt{-1}G_2(z) - \Gamma), \ \ \forall \gamma \in \mathrm{SL}_2(\mathbb{Z})$, so $\sqrt{-1}G_2(z) - \Gamma$ is a modular form of weight 2, must be 0.

Thus there exists only one holomorphic modular connection on $\mathbb{H}$. We can't find second holomorphic derivation of a modular form.

2. Apply to Siegel modular forms

2.1. The Levi-Civita connection

The Siegel upper half plane of degree g is defined to be the $g(g+1)/2$ dimensional open complex variety

$$\mathbb{H}_g := \{Z = X + \sqrt{-1}Y \in M(g, \mathbb{C}) \mid Z^t = Z, Y > 0\}.$$

Write $Z = (Z_{ij})$. Set $\Omega = \{(i, j) \mid 1 \leq i \leq j \leq g\}$ with the dictionary order. Then $\{dZ_{ij}, d\bar{Z}_{ij} \mid (i, j) \in \Omega\}$ is a series of coordinates on $\mathbb{H}_g$. The symplectic group of degree g over $\mathbb{R}$ is the group

$$\mathrm{Sp}(2g, \mathbb{R}) = \left\{M \in GL(2g, \mathbb{R}) \,\middle|\, MJM^t = J\right\},$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

The Siegel modular group $\Gamma_g := \mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{R})$ acts on $\mathbb{H}_g$ by the rule:

$$\gamma(Z) := (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathbb{H}_g, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z}).$$

By computing the connection coefficients Γ_{ij}^k by the formula, we can determine the Levi-Civita connection associated to the invariant metric $ds^2 = \mathrm{Tr}(Y^{-1}dZ \cdot Y^{-1}d\bar{Z})$ on the Siegel upper plane $\mathbb{H}_g$. Actually we have got,

Lemma .1 *Let $(r,s) \in \Omega$. Then*

$$D(dZ_{rs}) = -\sqrt{-1}(dZ_{s1}, dZ_{s2}, \cdots, dZ_{sg})Y^{-1} \cdot (dZ_{r1}, dZ_{r2}, \cdots, dZ_{rg})^t.$$

Now we recall the definition of Siegel modular forms: A (classical) Siegel modular form of weight k (and degree g) is a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ such that

$$f(\gamma(Z)) = \det(CZ + D)^k f(Z)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$ (with the usual holomorphicity requirement at ∞ when $g = 1$).

Let $M_{2k}(\Gamma_g)$ denote the $\mathbb{C}$-vector space of (non-holomorphic) Siegel modular forms of weight $2k$. Using the Levi-Civita connection above, we get

Theorem .2 *Let $f \in M_{2k}(\Gamma_g)$. Then*

$$\det\left(\left[\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1}\right] f\right) \in M_{2gk+2}(\Gamma_g),$$

where $\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{ij}}{2} \cdot \frac{\partial}{\partial Z_{ij}}\right)_{g\times g}$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Furthermore, we have the following more general result.

Theorem .3 *For any symmetric* $g \times g$ *matrix* $G(Z) = (G_{ij}(Z))$ *consisting of* $\mathbb{C}^\infty$ *(or holomorphic) functions which satisfies, for any* $\gamma \in Sp(2g, \mathbb{Z})$ *in the form above,*

$$(CZ + D)^{-1}\gamma(G) = G \cdot (CZ + D)^t + 2C^t,$$

there exists a unique modular connection satisfying

$$D(dZ_{rs}) = -\sum_{i,j=1}^{g} G_{ij} dZ_{si} dZ_{rj},$$

and for $f \in M_{2k}(\Gamma_g)$, *we have* $\det\left(\left[\frac{\partial}{\partial Z} - kG\right] f\right) \in M_{2gk+2}(\Gamma_g)$.

Moreover there is at most one holomorphic $G(Z)$ *satisfying the transformation formula, which, if exists, would give the holomorphic derivation of Siegel modular forms.*

But we can't prove the existence of the holomorphic G.

H. Maass once constructed a non-holomorphic derivative operator of a Siegel modular form f of weight k by invariant differential operators

$$D_k f(Z) = \det(Y)^{\kappa-k-1} \det\left(\frac{\partial}{\partial Z}\right) [\det(Y)^{k+1-\kappa} f(Z)],$$

where $\kappa = (g+1)/2$ and the determinant of $\frac{\partial}{\partial Z}$ is taken first, and showed that the differential operator D_k sends M_k to M_{k+2}. Compared to our operator, D_k is linear with respect to f. Moreover, our operator is a combination of degree 1 partial derivatives of f, but D_k is a combination of degree g partial derivatives.

3. Apply to Jacobi forms

3.1. The invariant metric

Let's recall the theory of Jacobi forms first. For given positive integers n, m, let $\mathbb{H}_n$ be the Siegel upper half plane of degree n, and $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$, called the Siegel-Jacobi space. An element of $\mathbb{H}_{n,m}$ can be written as (Z, W) with $Z = Z^t = (z_{ij}) \in M_{n,n}, W = (w_{rs}) \in M_{m,n}$.

Let $\mathrm{Sp}(n, \mathbb{R})$ be the symplectic group of degree n, and define

$$H_{\mathbb{R}}^{(n,m)} := \{(\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu\lambda^t \text{ symmetric}\}.$$

It is called the Heisenberg group, and endowed with the multiplication law:

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda^t\mu' - \mu^t\lambda')$$

The Jacobi group of degree n, index m is defined to be $G^J := \mathrm{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$, endowed with the following multiplication law

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot\Big(M',(\lambda',\mu';\kappa')\Big)=\Big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\Big)$$

with $M,M' \in \mathrm{Sp}(n,\mathbb{R})$; $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. G^J acts on $\mathbb{H}_{n,m}$ by

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot(Z,W) = \Big(M\cdot Z,(W+\lambda Z+\mu)(CZ+D)^{-1}\Big)$$

A (holomorphic) Jacobi form f of weight k and index M, with M a positive definite half integer $m \times m$ matrix, is a (holomorphic) function on $\mathbb{H}_{n,m}$, which satisfies the translation law of :

$$\begin{aligned} f(g(Z,W)) &= \det(CZ+D)^k \\ &\times e^{2\pi\sqrt{-1}\mathrm{Tr}(MW(CZ+D)^{-1}CW^t - M(\lambda Z\lambda^t + 2\lambda W^t - \mu\lambda^t))} f(Z,W) \end{aligned}$$

for $g = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu, \kappa) \right) \in G^J$.

The set of Jacobi forms of weight k and index M is denoted by $J_{k,M}$, and the subset of holomorphic Jacobi forms are denoted by $J_{k,M}^{hol}$.

The G^J-invariant metric on the Siegel-Jacobi plane $\mathbb{H}_{n,m}$ was given by J.-H. Yang: For any two positive real numbers a and b, the following metric on $\mathbb{H}_{n,m}$

$$\begin{aligned} ds^2_{n,m;a,b} =& a\,\mathrm{Tr}(Y^{-1}dZ\,Y^{-1}d\overline{Z}) \\ & + b\{\mathrm{Tr}(Y^{-1}V^tV\,Y^{-1}dZ\,Y^{-1}d\overline{Z}) + \mathrm{Tr}(Y^{-1}(dW)^t d\overline{W}) \\ & - \mathrm{Tr}(V\,Y^{-1}dZ\,Y^{-1}(d\overline{W}^t)) - \mathrm{Tr}(V\,Y^{-1}d\overline{Z}\,Y^{-1}(dW)^t)\} \end{aligned}$$

is G^J-invariant.

3.2. The Levi-Civita connection

The idea to construct derivative operators is that if we view the Siegel-Jacobi space as a Riemann surface with an invariant metric under the Jacobi group, then the Levi-Civita connection associated to this metric sends invariant sections to invariant sections.

Now we determine the Levi-Civita connection. Let Y, V denote the imaginary parts of Z and W respectively, and write

$$dZ = (dz_{ij}) \quad \text{and} \quad dW = (dw_{ij}).$$

Theorem .4 *Let D be the Levi-Civita connection on the Riemann surface $\mathbb{H}_{n,m}$ associated to the invariant metric above. Then*

$$D(dZ) = \frac{\sqrt{-1}b}{2a}(dZ, dW^t)\begin{pmatrix} 2\frac{a}{b}Y^{-1} + Y^{-1}V^tVY^{-1} & -Y^{-1}V^t \\ -VY^{-1} & I \end{pmatrix}\begin{pmatrix} dZ \\ dW \end{pmatrix} \tag{1}$$

$$\begin{aligned} D(dW) =& \frac{\sqrt{-1}b}{2a}VY^{-1}(dZ, dW^t)\begin{pmatrix} Y^{-1}V^tVY^{-1} & -Y^{-1}V^t \\ -VY^{-1} & I \end{pmatrix}\begin{pmatrix} dZ \\ dW \end{pmatrix} \\ & + \sqrt{-1}dWY^{-1}dZ \end{aligned} \tag{2}$$

where $D(dZ) = (D(dz_{ij}))$ and $D(dW) = (D(dw_{ij}))$.

3.3. Some differential operators of Jacobi forms

In the case of the degree $n = 1$, for a Jacobi form f of weight k and index M, by taking the connection on the G^J-invariant form $h = fy^k \exp(-4\pi Mv^2/y)\overline{f}$, we get two raising operators X_+, Y_+ and two lowering operators X_-, Y_-, which generated all linear differential operators of the Jacobi forms when $m = 1$, and non-holomorphic heat operator for general m. These cover old results. For general m, we also get some new holomorphic or meromorphic operators. For example,

$$\text{If} \quad f \in J_{k,M}^{hol}, \text{ then } \quad \frac{\partial^2 f}{\partial w^2} - 8\pi M\sqrt{-1}\frac{\partial f}{\partial z} + 2M(1-2k)G_2 f \in J_{k+2,M}^{hol}.$$

As for higher degree cases of $n > 1$, we are not able to get linear operators. However we can define some operators in the determinant form, including both raising and lowering operators and the heat operator, generalizing the classical case.

Actually we get

Theorem .5 *Let $f \in J_{k,m}$ be a Jacobi form on $\mathbb{H}_{n,m}$. Then we have*

(a) If $n = m$, $D_1(f) = \det(\frac{\partial f}{\partial W} + 4\pi\sqrt{-1}(Y^{-1}V^t M)f) \in J_{nk+1,nM}$; $\delta_1(f) = \det(\frac{\partial f}{\partial \overline{W}} Y) \in J_{nk-1,nM}$.

If $n < m$, choose any n rows of the matrix above and take determinant, we still get Jacobi forms in $J_{nk+1,nM}$ and $J_{nk-1,nM}$ respectively.

(b) $L_M(f) = \det(-8\pi\sqrt{-1}\frac{\partial f}{\partial Z} + \frac{\partial}{\partial W}M^{-1}(\frac{\partial f}{\partial W})^t - 4\pi k f Y^{-1} + 2m\pi f Y^{-1}) \in J_{nk+2,nM}$;

$$D_2(f) = \det(\frac{\partial f}{\partial Z} - \frac{\sqrt{-1}k}{2} f Y^{-1} + 2\pi\sqrt{-1}Y^{-1}V^t M V Y^{-1} f + \frac{\partial f}{\partial W} V Y^{-1}) \in J_{nk+2,nM};$$

$$\delta_2(f) = \det(\frac{\partial f}{\partial \overline{Z}} Y^2 + \frac{\partial f}{\partial \overline{W}} V Y) \in J_{nk-2,nM}.$$

Besides these, we also obtain the following holomorphic operators like the Rankin-Cohen brackets.

Theorem .6 *Let* $f \in J^{hol}_{k_1,M_1}$ *and* $g \in J^{hol}_{k_2,M_2}$ *on* $\mathbb{H}_{n,m}$*, then*

(a)*If* $n = m$, $\det(M_2\frac{\partial f}{\partial W}g - M_1\frac{\partial g}{\partial W}f) \in J_{n(k_1+k_2)+1,nM_1M_2}$.

(b) *In general,*

$$\det \quad \Big(\quad -8\pi\sqrt{-1}\frac{\partial f}{\partial Z}g(m-2k_2) + \frac{\partial}{\partial W}M_1^{-1}(\frac{\partial f}{\partial W})^t g(m-2k_2)$$
$$+ \quad 8\pi\sqrt{-1}\frac{\partial g}{\partial Z}f(m-2k_1) - \frac{\partial}{\partial W}M_2^{-1}(\frac{\partial g}{\partial W})^t f(m-2k_1)\Big)$$

is a Jacobi form of weight $(k_1+k_2)+1$*, index* nM_1M_2*.*

Using these results, we further obtain the following explicit invariant differential operators:

There exists operator matrix $\Gamma_{-}\Gamma_{+}$ which is invariant differential operator matrix on $\mathbb{H}_{n,m}$, thus each of the (k, l) entries of this matrix is an invariant differential operator on $\mathbb{H}_{n,m}$. Also there are invariant differential operators $H^j, T^j_{k,l}, U_{k,l}, V_{k,l}$. We omit the construction of these operators.

The invariant differential operators are these operators which are invariant under the action of the Jacobi group G^J. The result above is known only in the case $n = 1$ before.

Thanks!