Jacobi forms of rational weight and rational index

Hiroki Aoki （青木 宏樹）

Tokyo University of Science

（東京理科大学 理工学部）

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Modular forms of half-integral weights

On the theory of elliptic modular forms, there are some important modular forms of weight $1/2$.

- The Jacobi θ functions
- The Dedekind η function

More generally,

- Lattice θ-functions have weight $n/2$.
- θ-functions can be defined as modular forms of several variables.
- By Borcherds product, that is a generalization of the Dedekind η-function, we can construct modular forms of several variables of half-integral weights.

Question

Are there any modular forms whose weights are neither integral nor half-integral?

Modular forms of rational weight

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Are there any modular forms whose weights are neither integral nor half-integral?

- Bannai, Koike, Munemasa, Sekiguchi (1999)
  Klein's icosahedral equation and modular forms, preprint.
  Modular forms of weight $k/5$
- Ibukiyama (2000)
  Modular forms of rational weights and modular varieties, Abh. Math. Semi. Univ. Hamburg.
  Modular forms of weight $(N-3)/2N \quad (N>3:\text{odd integer})$
- Shimura (Recently)
  There is no interesting modular form whose weight is neither integral nor half-integral.
  (in his essay in Japanese)

Easy example

Dedekind η-function

Dedekind η-function is a holomorphic function on complex upper half plane $\mathbb{H}$ defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \qquad \left(q := \mathbf{e}(\tau) := \exp\left(2\pi\sqrt{-1}\tau\right)\right).$$

$$\eta(\tau + 1) = \mathbf{e}\left(\frac{1}{24}\right)\eta(\tau) \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$$

$\Longrightarrow$ $\eta(\tau)$ is a modular form of weight $1/2$ (w.r.t. $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{Mp}_2(\mathbb{Z})$).

Definition

Definition

We say F is a modular form of weight $k \in \mathbb{Q}$, if there exists $r \in \mathbb{N}$ such that F^r is a modular form of weight $kr \in \mathbb{Z}$ in usual sense.

In this sense $\eta(\tau)$ is a modular forms of weight $1/2$ w.r.t. $\mathrm{SL}_2(\mathbb{Z})$, because $\eta(\tau)^{24} = \Delta(\tau)$ is an elliptic modular form of weight 12.

Because η has no zero, for $k \in \mathbb{Q}$, we can define

$$\eta(\tau)^{2k} := \exp\left(2k \operatorname{Log} \eta(\tau)\right),$$

which is a modular form of weight k.

Modular forms of rational weights w.r.t. $\mathrm{SL}_2(\mathbb{Z})$

The following proposition is already known. (maybe)

Proposition.

Let F be a modular form of weight $k \in \mathbb{Q}$ w.r.t. $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\eta(\tau)^{-4\left\{\frac{k}{2}\right\}} F(\tau)$$

is an modular form of weight $2\left[\frac{k}{2}\right] \in 2\mathbb{Z}$ w.r.t. $\mathrm{SL}_2(\mathbb{Z})$, where $\left[\frac{k}{2}\right]$ and $\left\{\frac{k}{2}\right\}$ are the integral part and the fractional part of $\frac{k}{2}$.
In other words,

$$F(\tau) \in \eta(\tau)^{4\left\{\frac{k}{2}\right\}} \mathbb{C}[e_4(\tau), e_6(\tau)].$$

For vector valued elliptic modular forms w.r.t. an arbitrary discrete subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the dimension formula is known.

E. Freitag, *Dimension formulae for vector valued automorphic forms*, preprint, 2012.

Our motivation

There is no (non-trivial) example of modular forms of several variables whose weight is not integral or half-integral weight.

Question

Are there any modular forms of several variables whose weights are not integral or half-integral?

Remark: Borcherds product is a generalization of the Dedekind η-function, however, modular forms of several variables given by Borcherds product have zero.

Definition

Definition

We say F is a modular form of weight $k \in \mathbb{Q}$, if there exists $r \in \mathbb{N}$ such that F^r is a modular form of weight $kr \in \mathbb{Z}$ in usual sense.

Let $F : \mathbb{D} \to \mathbb{C}$ be a non-zero modular form of weight $k \in \mathbb{Q}$ w.r.t. Γ.

Then $J_F(\gamma; Z) := \dfrac{F(\gamma Z)}{F(Z)}$ $(Z \in \mathbb{D},\ \gamma \in \Gamma)$ satisfies the condition

$$J_F(\gamma_1\gamma_2; Z) = J_F(\gamma_1; \gamma_2 Z) J_F(\gamma_2; Z).$$

We have a group isomorphism

$$\Gamma \cong \Gamma_F := \left\{ (\gamma, J_F(\gamma; Z)) \in \Gamma \times \mathrm{Hol}(\mathbb{D}, \mathbb{C}^\times) \mid \gamma \in \Gamma \right\}.$$

r-th cover of Γ

Let

$$\widetilde{\Gamma}(k,r) := \left\{ (\gamma, J) \in \Gamma \times \mathrm{Hol}(\mathbb{D}, \mathbb{C}^\times) \,\middle|\, \gamma \in \Gamma,\ \begin{array}{c} J^r \text{ is an automorphic factor} \\ \text{of weight } kr \text{ w.r.t. } \gamma \end{array} \right\}.$$

Then Γ_F is a subgroup of $\widetilde{\Gamma}(k,r)$.

For $\widetilde{\gamma} = (\gamma, J) \in \widetilde{\Gamma}(k,r)$, define

$$(F|\widetilde{\gamma})(Z) := J(Z)^{-1} F(\gamma Z), \qquad \chi_F(\widetilde{\gamma}) := \frac{(F|\widetilde{\gamma})(Z)}{F(Z)}.$$

Then χ_F is a character of $\widetilde{\Gamma}$ and satisfies

$$\chi_F(\widetilde{\gamma})^r = 1, \quad \chi_F(id, \mathbf{e}(x)) = \mathbf{e}(-x).$$

Our strategy and our object

To find out modular forms of rational weights, we execute the following strategy:

1. Find a character of $\widetilde{\Gamma}(k,r)$ satisfying two conditions: $\chi(\widetilde{\gamma})^r = 1, \quad \chi(id, \mathbf{e}(x)) = \mathbf{e}(-x)$.
2. Find F such that $F|\widetilde{\gamma} = \chi(\widetilde{\gamma})F$.

We will apply this strategy to Jacobi forms, because Jacobi forms are not only the simplest modular forms of several variables but also the Fourier coefficients of many kinds of modular forms of several variables, for example Siegel modular forms and modular forms of $\mathrm{O}(2,n)$.

Jacobi group

The (full modular) Jacobi group is defined by

$$\mathrm{SL}_2(\mathbb{Z})^J := \left\{ (\gamma, \mathbf{u}) \;\middle|\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \mathbf{u} = (x, y) \in \mathbb{Z}^2 \right\},$$

with multiplication

$$(\gamma_1, \mathbf{u}_1) \cdot (\gamma_2, \mathbf{u}_2) := (\gamma_1\gamma_2, \mathbf{u}_1\gamma_2 + \mathbf{u}_2).$$

$\mathrm{SL}_2(\mathbb{Z})^J$ acts on $\mathbb{H} \times \mathbb{C}$ by

$$(\gamma, \mathbf{u})(\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + x\tau + y}{c\tau + d} \right).$$

The action of Jacobi group

Let $k, m \in \mathbb{Z}$. (k : weight, m : index)

For a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ and $(\gamma, \mathbf{u}) \in \mathrm{SL}_2(\mathbb{Z})^J$, define

$$(\varphi|\gamma)(\tau, z) := (c\tau + d)^{-k} \mathbf{e}\left(\frac{-mcz^2}{c\tau + d}\right) \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

$$(\varphi|\mathbf{u})(\tau, z) := \mathbf{e}(m(x^2\tau + 2xz))\varphi(\tau, z + x\tau + y)$$

and

$$\big(\varphi|(\gamma, \mathbf{u})\big)(\tau, z) := ((\varphi|\gamma)|\mathbf{u})(\tau, z).$$

$$\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{u} = (x, y)\right)$$

Weak Jacobi forms

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
The Jacobi group of Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})^J$ defined by

$$\Gamma^J := \{(\gamma, \mathbf{u}) \mid \gamma \in \Gamma,\ \mathbf{u} \in \mathbb{Z}^2\}.$$

Let φ be a Γ^J-invariant holomorphic function, namely, φ satisfies

$$\big(\varphi|(\gamma, \mathbf{u})\big)(\tau, z) = \varphi(\tau, z) \quad (\text{ for any } (\gamma, \mathbf{u}) \in \Gamma^J).$$

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, φ has a Fourier expansion

$$(\varphi|\gamma)(\tau, z) = \sum_{n,l} c_\gamma(n, l)\mathbf{e}(n\tau + lz).$$

We say φ is a weak Jacobi form of weight k and index m w.r.t. Γ^J, if φ is a Γ^J-invariant holomorphic function and if $c_\gamma(n, l) = 0$ for $n < 0$.

Examples

We denote the $\mathbb{C}$-vector space of all Jacobi forms of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ w.r.t. Γ^J by $\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J)$.

The following Jacobi forms were constructed by Eichler and Zagier.

$$\varphi_{-2,1}(\tau,z) = \zeta^{-1}(1-\zeta)^2 \prod_{j=1}^{\infty} \frac{\left(1-q^j\zeta\right)^2\left(1-q^j\zeta^{-1}\right)^2}{\left(1-q^j\right)^4} \in \mathbb{J}^{\mathrm{w}}_{-2,1}(\mathrm{SL}_2(\mathbb{Z})^J)$$

$$\varphi_{-1,2}(\tau,z) = \zeta^{-1}\left(1-\zeta^2\right) \prod_{j=1}^{\infty} \frac{\left(1-q^j\zeta^2\right)\left(1-q^j\zeta^{-2}\right)}{\left(1-q^j\right)^2} \in \mathbb{J}^{\mathrm{w}}_{-1,2}(\mathrm{SL}_2(\mathbb{Z})^J)$$

$$\varphi_{0,1}(\tau,z) = \frac{12}{\left(2\pi\sqrt{-1}\right)^2}\,\varphi_{-2,1}(\tau,z)\,\wp(\tau,z) \in \mathbb{J}^{\mathrm{w}}_{0,1}(\mathrm{SL}_2(\mathbb{Z})^J),$$

where $q := \mathbf{e}(\tau) := \exp\left(2\pi\sqrt{-1}\tau\right)$ and $\zeta := \mathbf{e}(z)$.

Some basic facts

Jacobi forms were first introduced by Eichler and Zagier in their book 'The theory of Jacobi forms' in 1985. The showed some basic facts on Jacobi forms and mentioned about the structure theorem of (weak) Jacobi forms.

Proposition. (Eichler, Zagier (1985))

Let φ be a weak Jacobi form of weight k and index m w.r.t. Γ^J. (Here we assume $k, m \in \mathbb{Z}$.)

1. If $m < 0$, $\varphi = 0$.
2. If $m = 0$, φ is an elliptic modular form of weight k w.r.t. Γ.

Proposition. (Eichler, Zagier (1985))

$$\dim \mathbb{J}^{\mathrm{w}}_{0,1}(\mathrm{SL}_2(\mathbb{Z})^J) = \dim \mathbb{J}^{\mathrm{w}}_{-2,1}(\mathrm{SL}_2(\mathbb{Z})^J) = \dim \mathbb{J}^{\mathrm{w}}_{-1,2}(\mathrm{SL}_2(\mathbb{Z})^J) = 1$$

Structure theorem (Integral weights and indices)

We denote the bi-graded ring of weak Jacobi forms by

$$\mathbb{J}^{\mathrm{w}}_{\mathbb{Z},\mathbb{Z}}(\Gamma^J) := \bigoplus_{k\in\mathbb{Z},\ m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J)$$

and the graded ring of elliptic modular forms by

$$\mathbb{M}_{\mathbb{Z}}(\Gamma) := \bigoplus_{k\in\mathbb{Z}} \mathbb{M}_k(\Gamma) \quad \left(= \bigoplus_{k\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,0}(\Gamma^J) \right).$$

Theorem. (Eichler, Zagier (1985), Ibukiyama, A. (2005))

$$\mathbb{J}^{\mathrm{w}}_{\mathbb{Z},\mathbb{Z}}(\Gamma^J) := \mathbb{M}_{\mathbb{Z}}(\Gamma)[\varphi_{0,1}, \varphi_{-2,1}] \oplus \varphi_{-1,2}\mathbb{M}_{\mathbb{Z}}(\Gamma)[\varphi_{0,1}, \varphi_{-2,1}]$$

Jacobi groups of rational weights and indices

Now we consider Jacobi forms of fractional weights and indices.

Let $k, m \in \mathbb{Q}$ and $r \in \mathbb{N}$ such that $kr, mr \in \mathbb{Z}$.
By studying the character of

$$\widetilde{\Gamma^J} := \widetilde{\Gamma^J}(k, r)$$
$$:= \left\{ ((\gamma, \mathbf{u}), J) \in \Gamma^J \times \mathrm{Hol}(\mathbb{H} \times \mathbb{C}, \mathbb{C}^\times) \;\middle|\; \begin{array}{c} \gamma \in \Gamma,\ \mathbf{u} \in \mathbb{Z}^2, \\ J^r \text{ is an automorphic factor} \\ \text{of weight } kr \text{ and index } mr \\ \text{w.r.t. } \gamma^J. \end{array} \right\},$$

we have some results on Jacobi forms of rational weights and indices.

Some properties

Here we assume $k, m \in \mathbb{Q}$.

Proposition.

Let φ be a weak Jacobi form of weight k and index m w.r.t. Γ^J.
If φ is not zero, then $2m \in \mathbb{Z}$ and $m \geq 0$. Moreover, if $m = 0$, φ is an elliptic modular form of weight k w.r.t. Γ.

Proposition.

Let φ be a weak Jacobi form of weight k and index m w.r.t. Γ^J. Then there exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ such that

$$\begin{aligned}\varphi(\tau, z) =& \varepsilon_1 \mathbf{e}(m(\tau + 2z))\varphi(\tau, z + \tau)\\ =& \varepsilon_2 \varphi(\tau, z + 1)\\ =& \varepsilon_3 \varphi(\tau, -z)\end{aligned}$$

Jacobi theta function

Jacobi theta function

$$\theta_{a,b}(\tau, z) := \sum_{n\in\mathbb{Z}} \mathbf{e}\left(\frac{1}{2}\left(n+\frac{1}{2}a\right)^2\tau + \left(n+\frac{1}{2}a\right)\left(z+\frac{1}{2}b\right)\right)$$

$$(\ a, b \in \{0,1\}\ )$$

is a (weak) Jacobi form of weight $1/2$ and index $1/2$.

Let

$$\varphi_A(\tau, z) := \frac{2\mathbf{e}\left(\frac{1}{24}\right)\eta(\tau)\theta_{0,0}(\tau, z)}{\eta\left(\frac{\tau+1}{2}\right)^2} \qquad \varphi_B(\tau, z) := \frac{2\eta(\tau)\theta_{0,1}(\tau, z)}{\eta\left(\frac{\tau}{2}\right)^2}$$

$$\varphi_C(\tau, z) := \frac{\eta(\tau)\theta_{1,0}(\tau, z)}{\eta(2\tau)^2} \qquad \varphi_D(\tau, z) := \frac{\theta_{1,1}(\tau, z)}{\eta(\tau)^3}$$

Examples

Proposition.

$\varphi_A, \varphi_B, \varphi_C$ and φ_D are weak Jacobi forms (of integral weights).

φ	k	m	ϵ_1	ϵ_2	ϵ_3	Γ
φ_A	0	$1/2$	1	1	1	Γ_A
φ_B	0	$1/2$	-1	1	1	Γ_B
φ_C	0	$1/2$	1	-1	1	$\Gamma_C = \Gamma_0(2)$
φ_D	-1	$1/2$	-1	-1	-1	$\mathrm{SL}_2(\mathbb{Z})$

Here

$$\Gamma_A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Gamma_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We remark $\varphi_{-2,1} = \varphi_D^2$ and $\varphi_{-1,2} = \varphi_A \varphi_B \varphi_C \varphi_D$.

Our Result (1)

For $k, m \in \mathbb{Q}$, We denote by

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; \varepsilon_1, \varepsilon_2, \varepsilon_3)$$

the set of all weak Jacobi forms of weight k and index m w.r.t. Γ^J with conditions $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

Theorem. (A.)

Let $k \in \mathbb{Q}$, $m \in \mathbb{Z}$ and $\varphi \in \mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; 1, 1, 1)$. Then there exist rational weights elliptic modular forms $f_0, f_1, \ldots, f_m$ of weight $k, k+2, \ldots, k+2m$ w.r.t. Γ such that

$$\varphi(\tau, z) = \sum_{j=0}^{m} f_j(\tau)\varphi_{-2,1}(\tau, z)^j \varphi_{0,1}(\tau, z)^{m-j}.$$

Our Results (2)

The case $m \in \mathbb{Z}$:

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,1,-1) = \varphi_{-1,2}\, \mathbb{J}^{\mathrm{w}}_{k+1,m-2}(\Gamma^J;1,1,1)$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,-1,1) = \begin{cases} \varphi_A\varphi_C\, \mathbb{J}^{\mathrm{w}}_{k,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,-1,-1) = \begin{cases} \varphi_B\varphi_D\, \mathbb{J}^{\mathrm{w}}_{k+1,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma_B) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;-1,1,1) = \begin{cases} \varphi_A\varphi_B\, \mathbb{J}^{\mathrm{w}}_{k,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;-1,1,-1) = \begin{cases} \varphi_C\varphi_D\, \mathbb{J}^{\mathrm{w}}_{k,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma_C) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;-1,-1,1) = \begin{cases} \varphi_B\varphi_C\, \mathbb{J}^{\mathrm{w}}_{k,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;-1,-1,-1) = \begin{cases} \varphi_A\varphi_D\, \mathbb{J}^{\mathrm{w}}_{k,m-1}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma_A) \\ \{0\} & (\text{otherwise}) \end{cases}$$

Our Results (3)

The case $m \in \mathbb{Z} + 1/2$:

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,1,1) = \begin{cases} \varphi_A \, \mathbb{J}^{\mathrm{w}}_{k,m-1/2}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma_A) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,1,-1) = \begin{cases} \varphi_B\varphi_C\varphi_D \, \mathbb{J}^{\mathrm{w}}_{k+1,m-3/2}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,-1,1) = \begin{cases} \varphi_C \, \mathbb{J}^{\mathrm{w}}_{k,m-1/2}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma_C) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J;1,-1,-1) = \begin{cases} \varphi_A\varphi_B\varphi_D \, \mathbb{J}^{\mathrm{w}}_{k+1,m-3/2}(\Gamma^J;1,1,1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

Our Results (4)

The case $m \in \mathbb{Z} + 1/2$: (cont.)

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; -1, 1, 1) = \begin{cases} \varphi_B \, \mathbb{J}^{\mathrm{w}}_{k,m-1/2}(\Gamma^J; 1, 1, 1) & (\Gamma \subset \Gamma_B) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; -1, 1, -1) = \begin{cases} \varphi_A \varphi_C \varphi_D \, \mathbb{J}^{\mathrm{w}}_{k+1,m-3/2}(\Gamma^J; 1, 1, 1) & (\Gamma \subset \Gamma(2)) \\ \{0\} & (\text{otherwise}) \end{cases}$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; -1, -1, 1) = \varphi_A \varphi_B \varphi_C \, \mathbb{J}^{\mathrm{w}}_{k,m-3/2}(\Gamma^J; 1, 1, 1)$$

$$\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^J; -1, -1, -1) = \varphi_D \, \mathbb{J}^{\mathrm{w}}_{k+1,m-1/2}(\Gamma^J; 1, 1, 1)$$

Thank you

Thank you for your kind attention.

BOCCHAN and MADONNACHAN, the mascots of Tokyo University of Science