Computing L-values and Petersson products via algebraic and p-adic modular forms

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p-adic modular forms

They were invented by J.-P.Serre [Se73] as limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The ring $\mathcal{M}_p$ of such forms contains $\mathcal{M} = \oplus_{k\geq 0}\mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$.
On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form. Let $\mathcal{N}$ be the ring of such forms over $\mathbb{Z}$.
Therefore

$$\tilde{E}_2|_{S=0} = E_2$$

is a p-adic modular form.
Elements of the ring $\mathcal{M}^\sharp = \mathcal{N}|_{S=0}$ are quasimodular forms. These phenomena are quite general and can be used in computations and proofs.

Using algebraic and p-adic modular forms in computations

There are several methods to compute various L-values attachted to Siegel modular forms using Petersson products of holomorphic and nearly-holomorphic Siegel modular forms :
the Rankin-Selberg method,
the doubling method (pull-back method).
A well-known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_k^n(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ a Dirichlet character.
Theorem (the case of even genus n (Courtieu-A.P.), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \leq s \leq k-n$ through the Petersson product $\langle f, \theta_{T_0} \delta^r E \rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0.
Theorem (the general case (by Boecherer-Schmidt), via the doubling method) uses Boecherer–Garrett–Shimura identity (a pull-back formula)

A pull-back formula

allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l+2s,\chi)D(l+2s-n,f,\chi)f = \langle f(w), E^{2n}_{l,\nu,\chi,s}(\mathrm{diag}[z,w])\rangle_w.$$

Here $k = l+\nu$, $\nu \geq 0$, $\Lambda(l+2s,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, $E^{2n}_{l,\nu,\chi,s}$ a higher twist of a Siegel-Eisenstein series on $(z,w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [Boe-Schm]).

A p-adic construction uses congruences for the L-values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms.

We indicate a new approach of computing the Petersson products and L-values, using an injection of algebraic nearly holomorphic modular forms into p-adic modular forms.

Applications to families of Siegel modular forms are given.

Explicit two-parameter families are constructed.

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], J. reine angew. Math., [Ich13]

allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into *p-adic modular forms*.

Via the Fourier expansions, the image of this injection is represented by certain quasimodular holomorphic forms like $E_2 = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$, with algebraic Fourier expansions.

This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L-values, which we would like to develop here.

This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms.

Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$-valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression

$$f(Z) = \sum_T P_T(S) q^T,$$

where T run through half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S.

Formal Fourier expansions

Algebraically we may use the notation

$$q^T = \exp(2\pi i \,\mathrm{tr}(TZ)) = \prod_{i=1}^{n} q_{ii}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ij}}$$
$$\in \mathbb{C}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$$

(with $q_{ij} = \exp(2\pi(\sqrt{-1}Z_{i,j})))$.
The elements q^T form a multiplicative semi-group so that $q^{T_1} \cdot q^{T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion over an arbitrary ring A via elements of the semi - group algebra $A[\![q^{B_n}]\!]$.
Namely, $f \in S_e(Sym^2(A^n), A[\![q^{B_n}]\!]^d)$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in Sym^2(A^n)$ of order n with vector values in $A[\![q^{B_n}]\!]^d$.

Holomorphic projection of nearly-holomorphic Siegel modular forms

Recall a passage from nearly holomorphic to holomorphic Siegel modular forms preserving the Petersson product with a given $f \in \mathcal{S}_k^n$. For an algebra homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$ over k, denote by $\mathcal{N}_\rho(k)$ the k-vector space of all $\mathbb{C}^d$-valued smooth functions which are nearly holomorphic over k with ρ-automorphic condition for $\Gamma(N)$. The elements of $\mathcal{N}_\rho(k)$ are nearly holomorphic Siegel modular forms over k of weight ρ, degree n, and level N.

Let $\rho = \det^{\otimes k} \otimes \rho_0$. By a structure theorem of Shimura (Prop. 14.2 at p.109 of [Sh00]), provided that k is large enough, for $h \in \mathcal{N}_\rho(k)$, $\boxed{h = \mathfrak{A}_{k,\rho_0}(h) + \Delta,}$ where $\mathfrak{A}_{k,\rho_0}(h) \in \mathcal{M}_\rho(k)$ is a holomorphic function and Δ is a finite sum of images of certain holomorphic functions under differential operators of Maass-Shimura type. Analytically $\mathfrak{A}_{k,\rho_0}(h)$ is the "holomorphic projection" of h.

Using Fourier expansions as p-adic modular forms

A method of computing with arithmetical nearly-holomorphic Siegel modular forms is based on the use of Ichikawa's mapping

$\iota_p : \mathcal{N}\rho \to \mathcal{M}p, \rho \overset{F_c}{\hookrightarrow} (\mathcal{R}_{g,p})^d$, where F_c is the Fourier expansion at a cusp c,

$$\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}.$$

Then the poynomial Fourier expansion of a nearly holomorphic form

$$f(Z) = \sum_T a_T(S) q^T \in \mathcal{N}\rho(\overline{\mathbb{Q}}),$$

over $\overline{\mathbb{Q}}$ becomes the Fourier expansion of an algebraic p-adic form over $i_p(\overline{\mathbb{Q}}) \subset \mathbb{C}_p$, whose Fourier coefficients can be obtained using Ichikawa's approach in [Ich13] by putting $S = 0$:

$$f \mapsto F_c(\iota_p(f)) = \sum_T a_T(0) q^T \in F_c(\mathcal{M}p, \rho).$$

Example. $f = \tilde{E}_2 = E_2 - \frac{3}{\pi y} = -12S + 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$ gives the p-adic modular form $F_c(\iota_p(f)) = E_2 = \tilde{E}_2|_{S=0}$ over $\mathbb{Z}$, which is also a quasimodular form of weight 2.
There are nice relations like $D(E_2) = \frac{1}{12}(E_2^2 - E_4)$ [MaRo5]

Computing the Petersson products

The Petersson product $h(Z) = \sum_T b_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by a given modular form $f(Z) = \sum_T a_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ gives a linear form

$$\ell_f : h \mapsto \frac{\langle f, h \rangle}{\langle f, f \rangle}$$

defined over a subring $R \subset \bar{\mathbb{Q}}$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients.

$$\ell_{T_i} : h \mapsto b_{T_i} \quad (i = 1, \ n).$$

It follows that $\ell_f(h) = \sum_i l_i b_{T_i}$.

How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L-function $L_f^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" (s_j,χ_j) one has an integral representation

$\boxed{L_f^*(s_j,\chi_j) = \langle f, h_j \rangle}$ with all $h_j = \sum_T b_{j,T} q^T \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$.

We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^* \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \frac{L_f^*(s_j,\chi_j)}{\langle f, f \rangle} \equiv 0 \mod p^N.$$

for any choice of $\beta_j \in \bar{\mathbb{Q}}$. Here $k_j = s_j - s_0$ or $k_j = -s_j + s_0$, according that there is $s_0 = \min_j s_j$ or $s_0 = \max_j s_j$.

Using the above expression for $\ell_f(h_j) = \sum_j l_{i,j} b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j} l_{i,j} \beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

Reduction to a finite dimensional case

In order to prove the congruences

$$\sum_{i,j} l_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{M} = \mathcal{M}(\overline{\mathbb{Q}})$

$$\mathcal{M}(\overline{\mathbb{Q}}) := \bigcup_{m\geq 0} \mathcal{M}_k(Np^m, \overline{\mathbb{Q}}).$$

Starting from the functions h_j, we use their caracteristic projection $\pi = \pi^\alpha$ on the characteristic subspace $\mathcal{M}^\alpha$ (of generalized eigenvectors) associated to a non-zero eigenvalue α Atkin's U-operator on f which turns out to be of fixed finite dimension so that for all j, $\pi^\alpha(h_j) \in \mathcal{M}^\alpha$.

From holomorphic to nearly holomorphic and p-adic modular forms

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} \subset \mathcal{N}_\rho(\overline{\mathbb{Q}})$ (of *nearly holomorphic modular forms*).

Usually, h_j can be expressed through the functions $\delta^{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$.

Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} = \mathcal{N}(\overline{\mathbb{Q}})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\overline{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{N}_{k,r}(Np^m, \overline{\mathbb{Q}}) \text{ (of the depth } r).$$

to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms.

This step respects the Petersson products with a conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h \rangle = \alpha^{-m} \langle f^0, |U(p)^m h \rangle = \langle f^0, \pi^\alpha(h) \rangle.$$

From holomorphic to nearly holomorphic and p-adic modular forms (continued)

2) To apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \to \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p-adic Siegel modular forms. Assume algebraically,

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T,$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T q^T,$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasimodular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

How to compute with Siegel modular forms?

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p-adic). We consider modular forms defined over $\mathbb{Q}$, over a number field $\begin{matrix} k \subset \bar{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C} \\ k \subset \bar{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p \end{matrix}$ or over a ring $\mathcal{R}$, and attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, for simplicity, attached to an algebraic representation $\rho_k = \rho_0 \otimes \det^{\otimes k}$ (like in [BoeNa13]).

We may take $\mathcal{R} = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[\![T]\!], \cdots$, and treat these modular forms as certain formal Fourier expansions over $\mathcal{R}$.

Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$f = \sum_T a_T(S) q^T \in \mathcal{N}\rho(R),$$

over R, and denote by $\mathcal{N}_{\rho,r}(\Gamma, R)$ the R-module of all such forms. This module is locally-free of finite rank, that is, over the fraction field $F = Frac(R)$, it becomes a finite-dimensional F-vector space.

Types of modular forms

- $\mathcal{M}_\rho$ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$)
- $\mathcal{N}_\rho$ (nearly holomorphic vector-valued Siegel modular forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$))
- $\mathcal{M}^\sharp_\rho$ (quasi-modular vector-valued forms attached to ρ)
- $\mathcal{M}^\flat_\rho$ (algebraic p-adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $\mathcal{M}^\sharp_{\rho,r} = \kappa(\mathcal{N}_\rho) \subset \mathcal{R}^d_{n,\infty}$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0) q^T$, where $\mathcal{R}_{n,\infty} = \mathbb{C}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.
- $\mathcal{M}^\flat_{\rho,r}(R,\Gamma) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R,\Gamma))) \subset \mathcal{R}^d_{n,p}$, where $\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.

Let us fix the level Γ, the depth r, and a subring R of $\bar{\mathbb{Q}}$, then all the R-modules $\mathcal{M}_\rho(R,\Gamma)$, $\mathcal{N}_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\flat_{\rho,r}(R,\Gamma)$ are then locally free of finite rank.
In interesting cases, there is an inclusion $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma) \hookrightarrow \mathcal{M}^\flat_{\rho,r}(R,\Gamma)$.
If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p-adic modular form, see [Se73], p.211.
Question:Prove it in general! (after discussions with S.Boecherer and T.Ichikawa)

Review of the algebraic theory

Following [Ha81], consider the columns $Z_1, Z_2, \ldots, Z_n$ of Z and the $\mathbb{Z}$-lattice L_Z in $\mathbb{C}^n$ generated by $\{E_1, \ldots, E_n, Z_1, \ldots, Z_n\}$, where $E_1, \ldots, E_n$ are the columns of the identity matrix E. The torus $\mathcal{A}_Z = \mathbb{C}^n / L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \longrightarrow \mathbb{H}_n$ whose fiber over the point Z is $\mathcal{A}_Z$.

Let us consider the quotient space $\mathbb{H}_n/\Gamma(N)$ of the Siegel upper half space $\mathbb{H}_n$ of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \;\middle|\; \begin{matrix} A_\gamma \equiv D_\gamma \equiv 1_n \\ B_\gamma \equiv C_\gamma \equiv 0_n \end{matrix} \right\}$$

If $N > 3$, $\Gamma(N)$ acts without fixed points on $\mathcal{A} = \mathcal{A}_n$ and the quotient is a smooth algebraic family $\mathcal{A}_{n,N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$ defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of 1.

For positive integers n and N, $\mathcal{H}_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension n with symplectic level N structure.

De Rham and Hodge vector bundles

The fiber varieties $\mathcal{A}$ and $\mathcal{A}_{n,N}$ give rise to a series of vector bundles over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}(\mathbb{C})$.

Notations

- $\mathcal{H}^1_{DR}(\mathcal{A}/\mathbb{H}_n)$ and $\mathcal{H}^1_{DR}(\mathcal{A}_{n,N}/\mathcal{H}_{n,N})$ the relative algebraic De Rham cohomology bundles of dimension $2n$ over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$) respectively. Their fibers at $Z \in \mathbb{H}_n$ are $H^1 := \mathrm{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C})$ generated by α_j, β_j:

$$\alpha_i(\sum_j a_j E_j + b_j Z_j) = a_i, \ \ \beta_i(\sum_j a_j E_j + b_j Z_j) = b_i \ \ (i = 1, \cdots, n).$$

- $\mathcal{H}^1_\infty$ the C^∞ vector bundle associated to $\mathcal{H}^1_{DR}$ (over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$). It splits as a direct sum $\mathcal{H}^1_\infty = \mathcal{H}^{1,0}_\infty \otimes \mathcal{H}^{0,1}_\infty$ and induces the Hodge decomposition on the De Rham cohomology of each fiber.
- The summand $\omega = \mathcal{H}^{1,0}_\infty$ is the bundle of relative 1-forms for either $\mathcal{A}/\mathbb{H}_n$ or $\mathcal{A}_{n,N}/\mathcal{H}_{n,N}$. Let us denote by $\pi : \mathcal{A}_{n,N} \to \mathcal{H}_{n,N}$ the universal abelian scheme with 0-section s, and by the Hodge bundle of rank n defined as

$$\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}}) = s^*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}})$$

- The bundle of holomorphic 1-forms on the base $\mathbb{H}_n$ or on $\mathcal{H}_{n,N}$, is denoted Ω.

Algebraic Siegel modular forms

are defined as global sections of $\mathbb{E}_\rho$, the locally free sheaf on $\mathcal{H}_{n,N} \otimes R$ obtained from twisting the Hodge bundle $\mathbb{E}$ by ρ.

Definition. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-algebra. For an algebra homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$ over R, define algebraic Siegel modular forms over R as elements of $\mathcal{M}_\rho(R) = H_0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho)$, called of weight ρ, degree n, level N.

If $\rho = \det^{\otimes k} : \mathrm{GL}_n \to \mathbb{G}_m$, then elements of $\mathcal{M}_k(R) = \mathcal{M}_{\det^{\otimes k}}(R)$ are called of weight k. For $R = \mathbb{C}$, each $Z \in \mathbb{H}_n$, let $\mathcal{A}_Z = \mathbb{C}^n/(\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)$ be the corresponding abelian variety over $\mathbb{C}$, and $(u_1, ..., u_n)$ be the natural coordinates on the universal cover $\mathbb{C}^n$ of $\mathcal{A}_Z$. Then $\mathbb{E}$ is trivialized over $\mathbb{H}_n$ by $du_1, ..., du_n$, and $f \in \mathcal{M}\rho(\mathbb{C})$ is a complex analytic section of $\mathbb{E}_\rho$ on $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$. Hence f is a $\mathbb{C}^d$-valued holomorphic function on $\mathbb{H}_n$ satisfying the ρ-automorphic condition:

$$f(Z) = \rho(C_\gamma Z + D_\gamma)^{-1} \cdot f(\gamma(Z)) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \right),$$

because $\mathcal{A}_Z \xrightarrow{\sim} \mathcal{A}_{\gamma(Z)}; {}^t(u_1, ..., u_n) \mapsto (CZ + D)^{-1} \cdot {}^t(u_1, ..., u_n)$, and γ acts equivariantly on the trivialization of $\mathbb{E}$ over $\mathbb{H}_n$ as the left multiplication by $(CZ + D)^{-1}$.

Algebraic Fourier expansion

can be defined algebraically using an algebraic test object over the ring $\mathcal{R}_n{=}\mathbb{Z}[\![q_{11},\ldots,q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]]_{i,j=1,\cdots,n}$, where $q_{i,j}(1\le i,j\le n)$ are variables with symetry $q_{i,j}=q_{j,i}$.
Mumford constructs in [Mu72] an object represented over $\mathcal{R}_n$ as

$$(\mathbb{G}_m)^n/\langle (q_{i,j})_{i=1,\cdots,n} \big| 1\le j\le n\rangle, (\mathbb{G}_m)^n = \mathrm{Spec}(\mathbb{Z}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]).$$

For the level N, at each 0-dimensional cusp c on $\mathcal{H}^*_{n,N}$, this construction gives an abelian variety over

$$\mathcal{R}_{n,N} = \mathbb{Z}[1/N,\zeta_N][\![q_{11}^{1/N},\ldots,q_{nn}^{1/N}]\!][q_{ij}^{\pm 1/N}]_{i,j=1,\cdots,n}$$

with a symplectic level N structure, and $\omega_i = dx_i/x_i$ $(1\le i\le n)$ form a basis of regular 1-forms.
We may view algebraically Siegel modular forms as certain sections of vector bundles over $\mathcal{H}_{n,N}$. Using the morphism $\mathrm{Spec}(\mathcal{R}_{n,N})\to\mathcal{H}_{n,N}$, $\mathbb{E}$ becomes $(\mathcal{R}_{n,N}\otimes R)^n$ in the basis $\omega_1,\ldots,\omega_n$.

Fourier expansion map and q-expansion principle

For an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, $\mathbb{E}_\rho$ becomes in the above basis ω_i

$$\mathbb{E}_\rho \times_{\mathcal{H}_{n,N}\otimes R} \mathrm{Spec}(\mathcal{R}_{n,N} \otimes R) = (\mathcal{R}_{n,N} \otimes R)^d.$$

For an R-module M, the space of Siegel modular forms with coefficients in M of weight ρ is defined as $\mathcal{M}_\rho(M) = H^0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho \otimes_R M)$. Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \to (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M)^d$$

which is called the Fourier expansion map associated with c. According to [Ich13], Theorem 2, F_c satisfies the following

q-expansion principle:

If M' is a sub R-module of M and $f \in \mathcal{M}_\rho(M)$ satisfies that $F_c(f) \in (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M')^d$, then $f \in \mathcal{M}_\rho(M')$.

Differential operators on modular forms, [Sh00],[Ich13]

Let $S_e(\mathrm{Sym}^2(R^n), R^d)$ be the R-module of all polynomial maps of $\mathrm{Sym}^2(R^n)$ into R^d homogeneous of degree e. For a $\mathbb{C}^d$-valued smooth function f of $Z = (z_{ij})_{i,j} = X + \sqrt{-1}Y \in \mathbb{H}_n$, consider $S_1(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^d)$-valued smooth functions $(Df)(u)$, $(Cf)(u)$ $(u = (u_{ij})_{i,j} \in \mathrm{Sym}^2(\mathbb{C}^n))$ of $\mathbb{Z} \in \mathbb{H}_n$

$$(Df)(u) = \sum_{1\le i\le j\le n} u_{ij}\frac{\partial f}{\partial(2\pi\sqrt{-1}z_{ij})}, \quad (Cf)(u) = (Df)((Z-\overline{Z})u(Z-\overline{Z})),$$

Let $\rho\otimes\tau^e : \mathrm{GL}_n(R) = \mathrm{GL}(R^n) \to \mathrm{GL}(S_e(\mathrm{Sym}^2(R^n), R^d))$ be the following R-homomorphism

$$[(\rho\otimes\tau^e)(\alpha)(h)](u) := \rho(\alpha)h({}^t\alpha\cdot u\cdot\alpha),$$

for $\alpha \in \mathrm{GL}_n(R), h \in S_e(\mathrm{Sym}^2(R^n), R^d)$, $u \in \mathrm{Sym}^2(R^n)$.
Then define $S_e(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^n)$-valued analytic functions $C^e(f)$, D^e $C^e(f), D^e_\rho(f)$ of $Z \in \mathbb{H}_n$ inductively, so that

$$D^e_\rho(f) = (\rho\otimes\tau^e)(Z-\overline{Z})^{-1}C^e(\rho(Z-\overline{Z})f).$$

D^e_ρ coincides with $(2\pi\sqrt{-1})^{-e}$ times Shimura's differential operator; it acts on arithmetical nearly-holomorphic Siegel modular forms.

Arithmetical nearly-holomorphic Siegel modular forms

Let $f(Z) = \sum_T a_T(S) \cdot q^{T/N} \in \mathcal{N}^r_\rho(k)$ be a nearly holomorphic Siegel modular forms over k, of weight ρ, degree n, level N for is a subfield k of $\mathbb{C}$ containing ζ_N, $q^{T/N} = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)/N)$, so that f is a $\mathbb{C}^d$-valued smooth function of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$, satisfying ρ-automorphic condition for $\Gamma(N)$ for an algebraic homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, namely

$$f(\gamma(Z)) = \rho(C_\gamma Z + D_\gamma)f(Z) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \right),$$ where

$a_T(S) \in \mathbb{C}^d$ are vectors whose entries are polynomials over k of degree r of the entries of the symmetric matrix $S = (4\pi Y)^{-1}$.
According to [Sh00], Chapter III, 12.10, if f satisfies the ρ-automorphic condition for $\Gamma(N)$, then $D^e_\rho(f)(u)$ satisfies the $\rho \otimes \tau^e$- automorphic condition: $D^e_\rho : \mathcal{N}_\rho \to \mathcal{N}_{\rho\otimes\tau^e}$ (defined over $\bar{\mathbb{Q}}$).
If f is arithmetical, $D^e_\rho(f)(u)$ is arithmetica and can be expressed through the Gauss-Manin connection ([Ha81], p.96) $\nabla = 1 \otimes d$, $\nabla(du_i) = \sum_j \beta_j dZ_{ij}$, $\nabla : H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \otimes \Omega^1(\mathbb{H}_n)$, using $H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C}) \otimes \mathcal{O}_{\mathbb{H}_n}$. Recall that ∇ computes to which extent the sections du_i fail to have constant periods: $du_i = \alpha_i + \sum_j \beta_j Z_{ij}$. Also, ∇ can be algebraically defined.

Arithmeticity of Shimura's differential operator ([Ich12],[Ich13], [Ha81], §4, [Ka78])

Proposition (see 2.2 of [Ich13]). Let $\pi : \mathcal{A} \to \mathbb{H}_n$ be the analytic family of

$$\mathcal{A}_Z = \mathbb{C}^n/(\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)(Z \in \mathbb{H}_n).$$

Then the normalized Shimura's differential operator D_ρ^e is obtained from the composition

$$\mathbb{E}_\rho \to \mathbb{E}_\rho \otimes (\Omega^1_{\mathbb{H}_n})^{\otimes e} \to \mathbb{E}_\rho \otimes (\mathrm{Sym}^2(\pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})))^{\otimes e},$$

the first map is given by the Gauss-Manin connection $\nabla : H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \otimes \Omega^1_{\mathbb{H}_n}$ together with the projection onto $\mathbb{E} = H^{1,0}$ in the Hodge decomposition of $H^1_{DR}(\mathcal{A}/\mathbb{H}_n)$, $H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to \pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})$; the second map is given by the Kodaira-Spencer isomorphism

$$\Omega^1_{\mathbb{H}_n} \overset{\sim}{\to} \mathrm{Sym}^2(\pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})), \quad \frac{dq_{i,j}}{q_{i,j}} \leftrightarrow \omega_i\omega_j = du_i du_j (1 \le i,j \le n)$$

Computing with families of Siegel modular forms

Let $\Lambda = \mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra, and consider Serre's ring

$$\mathcal{R}_{n,\Lambda} = \Lambda[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}^{\pm 1}]_{i,j=1,\cdots,n}.$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{k,\chi} : \Lambda \to \mathbb{C}_p, \mathcal{R}^d_{n,\Lambda} \mapsto \mathcal{R}^d_{n,\mathbb{C}_p}, \text{ where } T \mapsto \chi(1+p)(1+p)^k - 1.$$

Definition (families of Siegel modular forms)

Let $\mathbf{f} \in \mathcal{R}^d_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,

$$\kappa_{k,\chi}(\mathbf{f}) \in \mathcal{M}_{\rho_k}((i_p(\bar{\mathbb{Q}})) \overset{F_c}{\hookrightarrow} \mathcal{R}^d_{n,\mathbb{C}_p}$$

is the Fourier expansion at c of a Siegel modular form over $\bar{\mathbb{Q}}$.
All such $\mathbf{f}$ generate the Λ-submodule $\mathcal{M}_{\rho_k}(\Lambda) \subset \mathcal{R}^d_{n,\Lambda}$ of Λ-adic Siegel modular forms of weight ρ.
In the same way, the Λ-submodule $\mathcal{M}^{\sharp}_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}$ of Λ-adic Siegel quasi-modular forms is defined.

Examples of families of Siegel modular forms

can be constructed via differential operators of Maass $\Delta = \det(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}})$, so that $\Delta q^T = \det(T)q^T$. Shimura's operator $\delta_k f(Z) = (-4\pi)^{-n}\det(Z-\bar{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\bar{Z})^{k-\frac{1+n}{2}+1}f(Z))$ acts on q^T using $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r\mathbb{C}^n)$ and its adjoint ρ_r^*:

$$\delta_k(q^T) = \sum_{l=0}^{n}(-1)^{n-l}c_{n-l}(k+1-\frac{1+n}{2})\mathrm{tr}({}^t\rho_{n-l}(S)\rho_l^*(T))q^T,$$

where $c_{n-l}(s) = s(s-\frac{1}{2})\cdots(s-\frac{n-l-1}{2})$, $S = (2\pi i(\bar{z}-z))^{-1}$.

- Nearly holomorphic Λ-adic Siegel-Eisenstein series as in [PaSE] can be produced from the pairs $(-s,\chi)$: if s is a nonpositive integer such that $k+2s > n+1$,

$$E_k(Z,s,\chi) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1}\delta_{k+2s}^{(-s)}(E_{k+2s}(Z,0,\chi)).$$

Examples of families of Siegel modular forms (continued)

- Ichikawa's construction: quasi-holomorphic (and p-adic) Siegel - Eisenstein series obtained in [Ich13] using the injection ι_p

$$\iota_p(\pi^{ns} E_k(Z,s,\chi)) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1} \sum_T \det(T)^{-s} b_{k+2s}(T) q^T,$$

  where
  $E_{k+2s}(Z,0,\chi) = \sum_T b_{k+2s}(T) q^T, k+2s > n+1, s \in \mathbb{Z}$.
- A two-variable family is for the parameters $(k+2s, s), k+2s > n+1, s \in \mathbb{Z}$ will be now constructed.

Normalized Siegel-Eisenstein series of two variables

Let us start with an explicit family described in [Ike01], [PaSE], [Pa91] as follows

$$\mathcal{E}_k^n = E_k^n(z)2^{n/2}\zeta(1-k)\prod_{i=1}^{[n/2]}\zeta(1-2k+2i) = \sum_T a_T(\mathcal{E}_k^n)q^T,$$

where for any non-degenerate matrice T of quadratic character ψ_T:

$a_T(\mathcal{E}_k^n)$

$$= 2^{-\frac{n}{2}} \det T^{k-\frac{n+1}{2}} M_T(k) \times \begin{cases} L(1-k+\frac{n}{2},\psi_T)C_T^{\frac{n}{2}-k+(1/2)}, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

(C_T= cond(ψ_T), $M_T(k)$ a finite Euler product over $\ell|\det(2T)$. Starting from the holomorphic series of weight $k > n+1$ and $s=0$, let us move to all points $(k+2s,s), k+2s > n+1, s\in\mathbb{Z}, s\leq 0$. Then Ichikawa's construction is applicable and it provides a two-variable family.

Examples of families of Siegel modular forms (continued)

- Ikeda-type families of cusp forms of even genus [PaIsr11] (reported in Luminy, May 2011). Start from a p-adic family

$$\varphi = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k) q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!],$$

  where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p-parameters $\alpha(2k) := \alpha_p(2k)$ are given by certain p-adic analytic functions $k \mapsto a_n(2k)$ for $(n, p) = 1$. The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where $L(F_{2n}(\varphi), St, s) = \zeta(s) \prod_{i=1}^{2n} L(\varphi, s + k + n - i)$. This sequence provide an example of a p-adic family of Siegel modular forms.
- Ikeda-Myawaki-type families of cusp forms of $n = 3$, [PaIsr11] (reported in Luminy, May 2011).
- Families of Klingen-Eisenstein series extended in [JA13] from $n = 2$ to a general case
  (reported in Journées Arithmétiques, Grenoble, July 2013).

Thank you!

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