Spherical designs, modular forms, and toy models for D. H. Lehmer's conjecture

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This talk is based on the following papers.

1. **Eiichi Bannai and Tsuyoshi Miezaki: Toy models for D. H. Lehmer's conjecture, J. Math. Soc. Japan, 62 (2010), 687-705,**

2. **Eiichi Bannai and Tsuyoshi Miezaki: Toy models for D. H. Lehmer's conjecture, II, in Quadratic and Higher Degree Forms, (ed. by K. Alladi, et al.), Developments in Mathematics Volume 31, 2013, pp 1–27,**

3. **E. Bannai, T. Miezaki, and V. A. Yudin: An elementary approach to toy models for D. H. Lehmer's conjecture (Russian), Izvestiya RAN: Ser. Mat.75:6 (2011) 3–16, (translation: Izv. Math. 75 (2011), 1093–1106.)**

D. H. Lehmer's conjecture.

$$\begin{aligned}
q &= e^{\pi i z}, \quad z \in \mathbb{H}, \\
\eta(z) &= q^{\frac{1}{12}} \prod_{m \geq 1} (1 - q^{2m}) \\
&= q^{\frac{1}{12}} (1 - q^2 - q^4 + q^{10} + \cdots), \\
\Delta_{24} &= \eta(z)^{24} = q^2 \prod_{m \geq 1} (1 - q^{2m})^{24} \\
&= q^2 - 24q^4 + 252q^6 - 1472q^8 + 4830q^{10} \\
&\quad -6048q^{12} - 16744q^{14} + \cdots \\
&= \sum_{m \geq 1} \tau(m) q^{2m}.
\end{aligned}$$

τ is called Ramunujan's τ function.

Lehmer's Conjecture (1947):

$$\tau(m) \neq 0, \quad \text{for all positive integers } m.$$

(Compare with $|\tau(p)| < 2p^{\frac{11}{2}}$, Ramanujan-Deligne)

Lehmer's conjecture is known to be true for the following cases:

$m < 3316799 \approx 3 \cdot 10^6$ (Lehmer, 1947)
$m < 214928639999 \approx 2 \cdot 10^{11}$ (Lehmer)
$m < 10^{15}$ (Serre, 1985)
$m < 22689242781695999 \approx 2 \cdot 10^{16}$ (Jordan-Kelly, 1999)
$m < 22798241520242687999 \approx 2 \cdot 10^{19}$
(Bosman, 2007, arXiv:0710.1237v1)

- **Lehmer's conjecture can be restated in terms of spherical designs (Venkov, de la Harpe, Pache, 2005(?)).**
- **The original Lehmer's conjecture is still difficult to prove.**
- **We consider similar and easier situations, and solve them.**

(We call these cases toy models, and solve these cases.)

Definition (Spherical t-designs)

Let $r > 0$.
$S^{n-1}(r) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = r^2\}$ ($\subset \mathbb{R}^n$)
A finite subset $X \subset S^{n-1}(r)$ is called a spherical t-design, if and only if

$$\frac{1}{|S^{n-1}(r)|}\int_{S^{n-1}(r)} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for any polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree $\leq t$.

Let $\mathrm{Harm}_i(\mathbb{R}^n)$ = the space of homogeneous harmonic polynomials of degree i.
Then $X \subset S^{n-1}(r)$ is a spherical t-design, if and only if $\sum_{x\in X} f(x) = 0$, for all homogemeous harmonic polynomials of degrees $1 \leq i \leq t$.

Examples of spherical t-designs.

- $t+1$ vertices of a regular $(t+1)$-gon make a t-design in $S^1(\subset \mathbb{R}^2)$.
- 4 vertices of a regular simplex make a 2-design in $S^2(\subset \mathbb{R}^3)$.
- 8 vertices of a regular cube make a 3-design in $S^2(\subset \mathbb{R}^3)$.
- 12 vertices of a regular icosahedron make a 5-design in $S^2(\subset \mathbb{R}^3)$.
- 240 roots of type E_8 make a 7-design in $S^7(\sqrt{2})(\subset \mathbb{R}^8)$.
- 196560 min. vectors of Leech lattice make an 11-design in $S^{23}(2)(\subset \mathbb{R}^{24})$.

It is known that spherical t-designs on S^{n-1} exist for any n and any t (Seymour-Zaslavsky, 1984). However, explicit constructions are very difficult for large t and $n \geq 3$.

Spherical t-designs which are obtained from shells of lattices.

Let L be an integral lattice in $\mathbb{R}^n$. For an integer m, the shell L_m is defined by:

$$L_m = \{x \in L \mid x \cdot x = m\} \quad (\subset S^{n-1}(\sqrt{m})).$$

We are interested in the properties of shells $X = L_m$ as spherical t-designs.

Theorem of Venkov (1984)

Let L be an even unimodular lattice in $\mathbb{R}^n$. (Then $n \equiv 0 \pmod 8$.) Moreover, let L be an extremal even unimodular lattice, i.e.,

$$\min\{x \cdot x \mid x \in L, x \neq 0\} = 2 + 2\left[\frac{n}{24}\right].$$

Then, for any m, $X = L_{2m}$ is a spherical

$$\begin{cases} 11\text{-design}, & \text{if } n \equiv 0 \pmod{24} \\ 7\text{-design}, & \text{if } n \equiv 8 \pmod{24} \\ 3\text{-design}, & \text{if } n \equiv 16 \pmod{24} \end{cases}$$

In particular, any shell $X = L_{2m}$ of the E_8 root lattice L is a 7-design.

Problem.

Is there any 8-design among the shells $X = L_{2m}$ of E_8-lattice L ?

(Note that a t-design is a t'-design if $t' \leq t$.)

Key Observation by Venkov:

For the E_8-lattice L, the shell L_{2m} is an 8-design, if and only if $\tau(m) = 0$,

where τ is the Ramanujan tau function.

So, D. H. Lehmer's conjecture is equivalent to the fact that there is no spherical 8-design among the shells of the E_8-lattice L.

Theorem (Hecke, Schoeneberg, 1930's)

For an even unimodular lattice L in $\mathbb{R}^n$, the theta series $\Theta_L(z)$ of lattice L is defined by:

$$\Theta_L(z) = \sum_{x \in L} e^{\pi i z (x \cdot x)} = \sum_{m \geq 0}^{\infty} |L_{2m}| q^{2m}.$$

For $P(x) \in \mathrm{Harm}_k(\mathbb{R}^n)$, the theta series $\Theta_{L,P}(z)$ with homogeneous harmonic polynomial $P(x)$ is defined by:

$$\Theta_{L,P}(z) = \sum_{x \in L} P(x) e^{\pi i z (x \cdot x)} = \sum_{m \geq 0}^{\infty} \Big(\sum_{x \in L_{2m}} P(x) \Big) q^{2m}.$$

Then, $\Theta_{L,P}(z)$ is a modular form of weight $\frac{n}{2} + k$ w.r.t. $SL(2, \mathbb{Z})$. Moreover, $\Theta_{L,P}(z)$ is a cusp form if $k \geq 1$.

Proof of Venkov's theorem for E_8-lattice L.

We need to show that

$$\sum_{x \in L_{2m}} P(x) = 0$$

for any $P \in \mathrm{Harm}_k(\mathbb{R}^8)$ with $1 \leq k \leq 7$. Since the result is obvious for odd k, we have only to show for $k = 2, 4, 6$. Now, let take such P. Then $\Theta_{L,P}(z)$ is a cusp form (w.r.t. $SL(2, \mathbb{Z})$) of weight $\frac{n}{2} + k = 4 + k = 6, 8,$ or 10. Since there is no non-zero cusp forms of weight 6, 8, and 10, we have:

$$\Theta_{L,P}(z) = \sum_{m \geq 0}^{\infty} \Big(\sum_{x \in L_{2m}} P(x) \Big) q^{2m} \equiv 0.$$

Thus, $\sum_{x \in L_{2m}} P(x) = 0$. QED.

Proof of Venkov's Key Observation.

Let L be the E_8-lattice in $\mathbb{R}^8$, and let $P \in \mathrm{Harm}_8(\mathbb{R}^8)$. Then, since the dimension of cusp form of weight $12(= \frac{n}{2} + k = 4 + 8)$ is 1, we have

$$\Theta_{L,P}(z) = \sum_{m \geq 0}^{\infty} (\sum_{x \in L_{2m}} P(x)) q^{2m} = c(P)\Delta_{24}(z)$$

where $\Delta_{24} = \sum_{m=1}^{\infty} \tau(m) q^{2m}$.
• Let L_{2m} not be an 8-design. Then there exists a $P \in \mathrm{Harm}_8(\mathbb{R}^8)$ such that $\sum_{x \in L_{2m}} P(x)(= c(P)\tau(m)) \neq 0$. So, $\tau(m) \neq 0$.
• Suppose $\tau(m) \neq 0$. We can easily see that $L_2(=$ roots of type $E_8)$ is not an 8-design. Hence, there exists $P \in \mathrm{Harm}_8(\mathbb{R}^8$ such that $\sum_{x \in L_2} P(x) \neq 0$. Hence $c(P) \neq 0$ (for this P.) Thus, $\sum_{x \in L_{2m}} P(x) = c(P)\tau(m) \neq 0$. Hence L_{2m} is not an 8-design.

QED.

Lehmer's attempt (1947) to try to prove $\tau(m) \neq 0$.

step 1.

(i) $\tau(mn) = \tau(m)\tau(n)$, if $(m, n) = 1$.

(ii) $\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1})$, for $p =$ prime.

(iii) ($|\tau(p)| < 2 \cdot p^{11}$, Ramanujan-Deligne) Put $2\cos\theta_p = \tau(p)p^{-\frac{11}{2}}$. Then

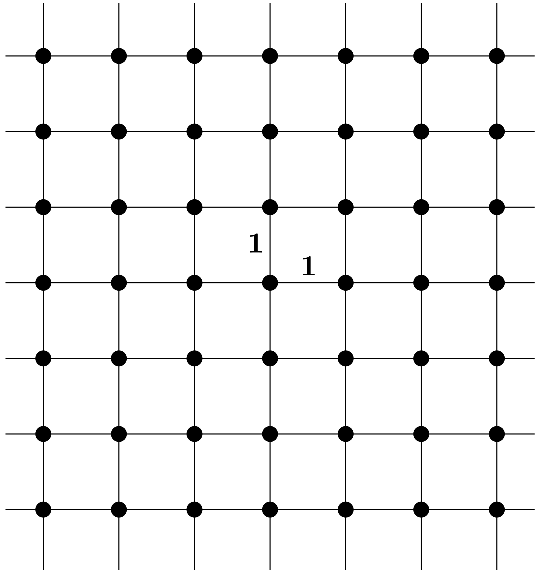
$$\tau(p^{\alpha}) = p^{\frac{11}{2}\alpha} \cdot \frac{\sin(\alpha+1)\theta_p}{\sin\theta_p}.$$

step 2. Let m be the smallest integer ($m \geq 1$) such that $\tau(m) = 0$. Then $m = p^{\alpha}$, and moreover, $m = p$ (i.e. $\alpha = 1$).

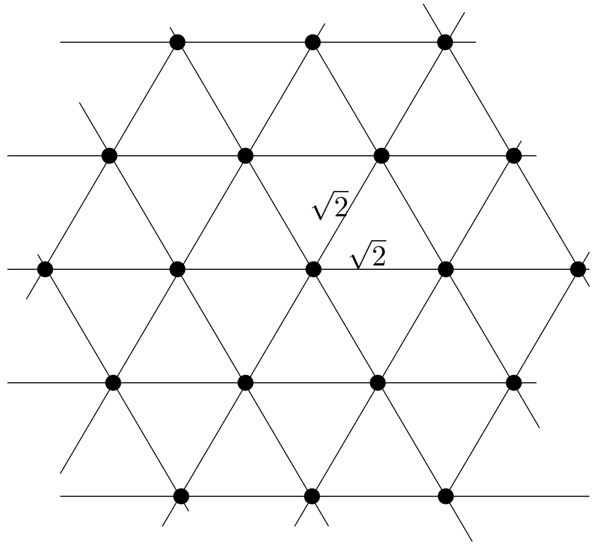
step 3. If $\tau(p) = 0$, $p =$ prime, then we get many congruence conditions w.r.t. several prime powers.
(So, he could get $\tau(p) \neq 0$, if $p < 3316799$, say.
But he could not prove it for general p.)

We consider toy models of the following two lattices.

(i) $\mathbf{L} = \mathbb{Z}^2 \subset \mathbb{R}^2$



(ii) $\mathbf{L} = \mathbb{A}_2 \subset \mathbb{R}^2$



- **Let $L = \mathbb{Z}^2$-lattice. Let $L_m = \{x \in L \mid x \cdot x = m\}$. Then all the non-empty shells L_m are 3-designs. (Can any of them be a 4-design?)**
- **Let $L = A_2$-lattice. Let $L_m = \{x \in L \mid x \cdot x = m\}$. Then all the non-empty shells L_m are 5-designs. (Can any of them be a 6-design?)**

Theorem (Bannai-Miezaki, 2010 [1])

(i) For the $\mathbb{Z}^2$-lattice L, no non-empty shell can be a 4-design.

(ii) For the A_2-lattice L, no non-empty shell can be a 6-design.

Sketch of Proof (for $\mathbb{Z}^2$-lattice L).

$\theta_3(z) = \sum_{m\in\mathbb{Z}} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots,$
$\Theta_L(z) = \theta_3(z)^2 = \sum_{m=0}^{\infty} r_2(m)q^m$
$= 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + \cdots.$

(Note that $r_2(m) \neq 0$, if and only if any prime $p \equiv 3 \pmod 4$ which divides m divides m exactly with even power.)

$\Theta_{L,P}(z) = c_1(P)\Delta_8(z)\theta_3(z)^2,$
where $\Delta_8(z) = \frac{1}{16}\theta_3(z)^4\theta_4(z)^4 = q - 8q^2 + 28q^3 + \cdots,$
and $P \in \mathrm{Harm}_4(\mathbb{R}^2)$
(Pache, 2005)

Now, let us set

$$\Delta_8(z)\theta_3(z)^2 = \sum_{m\geq 1} a(m)q^m.$$

$a(m)$ plays a similar role as $\tau(m)$ (for E_8-lattice), and we get:

- L_m is a 4-design, if and only if $a(m) = 0$.

We have

$$\Delta_8(z)\theta_3(z)^2 = \sum_{m \geq 1} a(m)q^m \in S_5(G(2), \chi)$$

and

$$\Delta_8(2z)\theta_3(2z)^2 = \sum_{m \geq 1} a(m)q^{2m} \in S_5(\Gamma_0(4), \chi_4),$$

the space of cusp forms of weight 5 w.r.t. $\Gamma_0(4)$ and a certain character χ_4.

$S_5(\Gamma_0(4), \chi_4)$, is of dimension 1, and so $\Delta_8(2z)\theta_3(2z)^2$ is a normalized Hecke eigenform. Then we have the following assertions.

step 1. (i) $\tau(mn) = \tau(m)\tau(n)$, if $(m, n) = 1$.
(ii) $\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - \chi_4 p^4 \tau(p^{\alpha-1})$, for $p =$ prime.
(iii) ($|\tau(p)| < 2 \cdot p^{-2}$.) Put $2\cos\theta_p = \tau(p)p^{-2}$. Then (for $p \equiv 1 \pmod 4$),

$$\tau(p^{\alpha}) = p^{2\alpha} \cdot \frac{\sin(\alpha+1)\theta_p}{\sin\theta_p}.$$

step 2. Let m be the smallest integer ($m \geq 1$) such that $r_2(m) \neq 0$ and $a(m) = 0$. Then $m = p^{\alpha}$, and moreover, $m = p$ (i.e. $\alpha = 1$) with $p \equiv 1 \pmod 4$.

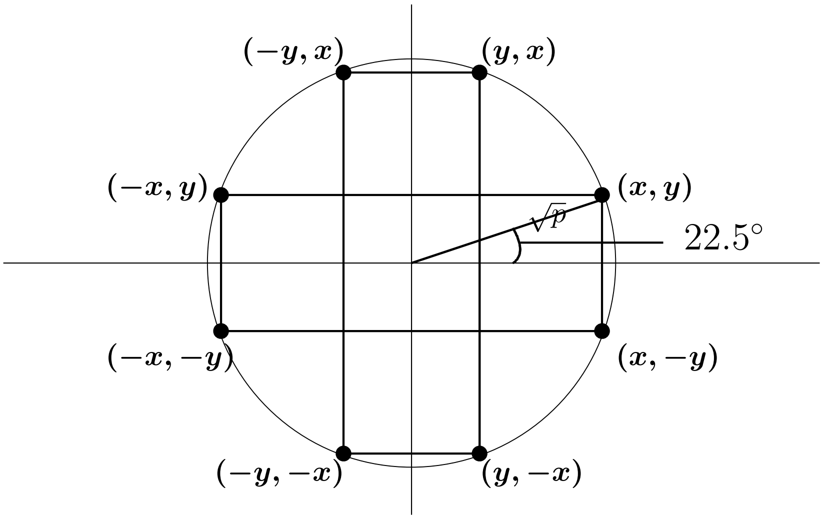
step 3. For such prime p, we get

$$r_2(p) = 8, \qquad \text{(i.e. } L_p = 8).$$

Then we can get a contradiction by considering some congruence properties.

An alternative combinatorial proof is also possible.

These 8 points of L_p must be as below. It is easy to see that in order that such 8 points form a 4-design, they must be the set of vertices of a regular 8-gon. However, since $\tan(22.5°) = \frac{y}{x}$ is not a rational number, we get a contradiction.



Proof for A_2-lattice is similar. We use $(\Gamma_0(3), \chi)$ instead of $(\Gamma_0(4), \chi_4)$, and at the last step, we use that $\tan(15°)$ is not a rational number.

More toy models.

Let $d > 0$ is a square free positive integer,
$K = \mathbb{Q}(\sqrt{-d})$,
$\mathcal{O} = \mathcal{O}_K =$ the set of algebraic integers of H

$$= \begin{cases} \mathbb{Z}[\sqrt{-d}], & \text{if } -d \equiv 2, 3 \pmod 4, \\ \mathbb{Z}[\frac{1+\sqrt{-d}}{2}], & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

$L = L_{\mathcal{O}} =$ the corresponding lattice in $\mathbb{R}^2$

$$= \begin{cases} \mathbb{Z}(1,0) + \mathbb{Z}(0, \sqrt{d}), & \text{if } -d \equiv 2, 3 \pmod 4, \\ \mathbb{Z}(1,0) + \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{d}}{2}), & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

(note that $L_{\mathcal{O}}$ is $\mathbb{Z}^2$-lattice if $d = 1$, and $L_{\mathcal{O}}$ is A_2-lattice if $d = 3$.)

Then we have the following results.

Theorem (Bannai-Miezaki, 2013 [2])

(i) Suppose that $d \in \{2, 7, 11, 19, 43, 67, 163\}$ (i.e. let $K = \mathbb{Q}(\sqrt{-d})$ has class number 1, and $d \neq 1$ and $\neq 3$). Then any (non-empty) shell of $L = L_{\mathcal{O}}$ is not a 2-design.

(ii) Suppose that $d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$ (i.e. $K = \mathbb{Q}(\sqrt{-d})$ has class number 2. Then any (non-empty) shell of $L = L_{\mathcal{O}}$ is not a 2-design.

Idea of the Proof.

We use the following known result:

K = algebraic number field over $\mathbb{Q}$,
Λ = a nontrivial ideal of $\mathcal{O}_K$,
$I(\Lambda)$ =the set of fractional ideals prime to Λ.
Let $\phi : I(\Lambda) \longrightarrow \mathbb{C}^*$
be a Hecke character of weight k. Then

$$\Psi_{K,\Lambda}(z) = \sum_{A} \phi(A) q^{N(A)} = \sum_{n=1}^{\infty} a(n) q^n$$

is a cusp form in

$$S_k(\Gamma_0(d_K \cdot N(\Lambda)), (\frac{-d_K}{\cdot}) \cdot \omega_\phi),$$

and moreover, $\Psi_{K,\Lambda(z)}$ is a Hecke eigenform, where A runs over the integral ideals prime to Λ, and $N(A)$ is the norm of ideal A.

We apply this for the case, $K = \mathbb{Q}(\sqrt{-d}), \Lambda = (1) = \mathcal{O}, k = 3$.

In the case of class number 1, we can see that $\Theta_{L_{\mathcal{O}},P}$ for an appropriate $P \in \mathrm{Harm}_2(\mathbb{R}^2)$, becomes a normalized Hecke eigenform $\Psi_{K,\Lambda}(z) = \sum_{m\geq 1} a(m)q^m$, with all the $a(m)$ integers.

In the case of class number 2, we can see that, if we take constants c_1 and c_2 appropriately, then $(c_1\Theta_{L_{\mathcal{O}},P} + c_2\Theta_{L_{\mathcal{O}'},P}) = \sum_{m\geq 1} a(m)q^m$, (where $P \in \mathrm{Harm}_2(\mathbb{R}^2)$ and $\mathcal{O}'$ is a non-principal ideal), becomes a normalized Hecke eigenform $\Psi_{K,\Lambda}(z) = \sum_{m\geq 1} a(m)q^m$, with all the $a(m)$ integers. So, a similar argument as before outlined as in step 1, step 2 and step 3 works.

However, if the class number is 3 or more, it seems difficult to find a normalized Hecke eigenform whose coefficients are all integers. (It seems that, for $d = 23$, the coefficients of Hecke eigenform cannot be even in a cyclotomic number field.) This is why we have difficulty in extending our result beyond the class number 2 case. although we believe that the conclusion (namely, there are no 2-designs among non-empty shells of $L_{\mathcal{O}_K}$) always holds for $d \neq 1, \neq 3$.

Speculations.

1 (i) For any (integral) lattice L in $\mathbb{R}^2$, no shell becomes a 6-design.
(Can you prove this?)
(Can you characterize the lattices in $\mathbb{R}^2$, which has a 4-design among shells?)

(ii) For any (integral) lattice L in $\mathbb{R}^3$, no shell becomes a 4-design.
(Can you prove this?)

(iii) For any (integral) lattice L in $\mathbb{R}^n$, no shell becomes an 11-design.
(Can you prove this?)

2 It is possible to prove toy models for Lehmer's conjecture for $L_{\mathcal{O}}$ for $K = \mathbb{Q}(\sqrt{-d})$ with class number 1, by another method, without using modular forms directly, (see Bannai-Miezaki-Yudin, 2011 [3].)

3 Anyway, it would be extremely interesting to prove toy models for lattices in higher dimensions.

Thank you very much