Explicit Theory of Automorphic Forms

Refined version of Ikeda's conjecture on the Period of the Hermitian Ikeda lift (and its application)

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1. Introduction

Γ : a modular group (symplectic group, unitary group etc.)
η : a character of Γ
$S_l(\Gamma, \eta)$: the space of cusp forms of weight l and character η for Γ
$\langle F, F \rangle$ the period (the Petersson norm) of F

Γ : a cogruence subgroup of $\mathrm{SL}_2(\mathbf{Z}), k \in \mathbf{Z}/2$
For a Hecke eigenform $g \in S_k(\Gamma, \eta)$, let $\widehat{g}$ be a certain "lift" of g to "another space of cusp forms", that is, let $\widehat{g}$ be a Hecke eigenform whose certain L-function is expressed in terms of certain L-functions related with g.

Problem A. Express $\dfrac{\langle \widehat{g}, \widehat{g} \rangle}{\langle g, g \rangle^e}$ in terms of certain L-values related with g.

Why is the above problem interesting and important ?

(1) The expression in Problem A (if exists) connects analytic objects(periods) with arithmetic objects(L-values).

(2) Sometimes one can prove the algebraicity of the above ratio by using such an expression in Problem A, and prime ideals appearing in that ratio give a congruence beteween $\widehat{g}$ and another cuspidal Hecke eigenform not coming from the lift.
(cf. Brown, Boecherer-Dummigan-Schulze-Pillot,Ibukiyama-Poor-Yuen-K. Klosin
K.)

Known results

(1) Zagier (1977): Doi-Naganuma lift
Let $\widehat{g}$ be the lift of a primitive form $g \in S_k(SL_2(\mathbf{Z}))$ to $S_k(SL_2(\mathcal{O}_K))$ with some real quadratic field K. Then

$$\frac{\langle \widehat{g}, \widehat{g} \rangle}{\langle g, g \rangle} \approx L(1, f, \mathrm{Ad}, \chi_K) \ (= L(k, f, Sym^2, \chi_K)).$$

(2) Kohnen and Skoruppa(1989): Saito-Kurokawa lift

(3) Kawamura-K(2007): Duke-Imamoglu-Ikeda lift (D-I-I lift)
Let $I_{2n}(g)$ be the D-I-I lift of a Hecke eigenform g in the Kohnen plus subspace $S_{k-n+1/2}(\Gamma_0(4))^+$ to $S_k(Sp_{2n}(\mathbf{Z}))$ and f the primitive form in $S_k(SL_2(\mathbf{Z}))$ corresponding to g under the Shimura correspondence. Then

$$\frac{\langle I_{2n}(g), I_{2n}(g) \rangle}{\langle g, g \rangle} \approx L(k+2n, f) \prod_{i=1}^{n} \zeta(2i) L(2i-1, f, \mathrm{Ad})$$

(4) K(2010): Answer to Problem A for the Hemitian Ikeda lift

(5) Boecherer,Dummigan,and Schulze-Pillot (2012): Answer to Problem A for the Yoshida lift

The aim of today's talk is to give a refined version of (4), and its adelic version.

2. Hermitian Ikeda lifts

K : an imaginary quadratic field with the discr. $-D$,
$\mathcal{O}$: the ring of integers in K
$\chi := \left(\frac{-D}{*}\right)$

S: $\mathbf{Q}$-algebra
$R_{K/\mathbf{Q}}(GL_m)(S) = GL_m(S \otimes_{\mathbf{Q}} K)$
$U(m,m)(S) := \{g \in GL_{2m}(S \otimes_{\mathbf{Q}} K) \mid g^* \begin{pmatrix} O_m & -1_m \\ 1_m & O_m \end{pmatrix} g = \begin{pmatrix} O_m & -1_m \\ 1_m & O_m \end{pmatrix}\}$
$g^* := {}^t\overline{g}$

G : alg. gp./$\mathbf{Q}$
$G_{\mathbf{A}}$: the adelization of G
$G_{\mathbf{h}}$: the finite part of $G_{\mathbf{A}}$

$K_p := K \otimes_{\mathbf{Q}} \mathbf{Q}_p, \ \mathcal{O}_p := \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p$

$\mathcal{K}^{(m)} := (U(m,m)_{\mathbf{h}} \cap \prod_p GL_{2m}(\mathcal{O}_p))U(m,m)(\mathbf{R}) \subset U(m,m)_{\mathbf{A}}$

$h = h_K$: the class number of K

$U(m,m)_{\mathbf{A}} = \bigsqcup_{i=1}^{h} U(m,m)(\mathbf{Q})\gamma_i \mathcal{K}^{(m)}$

$\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{*-1} \end{pmatrix}$ $t_i \in R_{K/\mathbf{Q}}(GL_m)_{\mathbf{h}}$ s.t. $t_1 = 1_m$ and

$R_{K/\mathbf{Q}}(GL_m)_{\mathbf{A}} = \bigsqcup_{i=1}^{h} GL_m(K) t_i \prod_p GL_m(\mathcal{O}_p) GL_m(\mathbf{C})$

$\Gamma_i^{(m)} = U(m,m)(\mathbf{Q}) \cap \gamma_i \mathcal{K}^{(m)} \gamma_i^{-1}$

Rem. $\Gamma_1^{(m)} = \{\gamma \in GL_m(\mathcal{O}) \mid g^* \begin{pmatrix} O_m & -1_m \\ 1_m & O_m \end{pmatrix} g = \begin{pmatrix} O_m & -1_m \\ 1_m & O_m \end{pmatrix}\}$

$\mathfrak{H}_m := \{Z \in M_m(\mathbf{C}) \mid \frac{Z - Z^*}{2\sqrt{-1}} > 0\}.$

$S_{(2l,-l)}(\mathit{\Gamma}_i^{(m)})$
$:= \{F : \mathfrak{H}_m \longrightarrow \mathbf{C} \mid F(\gamma(Z)) = j(\gamma, Z)^{2l}(\det \gamma)^{-l}F(Z)$for $\gamma \in \mathit{\Gamma}_i^{(m)}$
$+$ cuspidal condition$\}$
(the space of Hermitian cusp forms of weight $(2l, -l)$ for $\mathit{\Gamma}_i^{(m)}$)
$\langle F, F\rangle = [\mathit{\Gamma}_1^{(m)} : \mathcal{O}^*(\mathit{\Gamma}_1^{(m)} \cap \mathit{\Gamma}_i^{(m)})]^{-1} \int_{(\mathit{\Gamma}_1^{(m)} \cap \mathit{\Gamma}_i^{(m)})\backslash \mathfrak{H}_m} |F(Z)|^2 \det Y^{l-2m} dX dY$
(the period of F)

$\mathcal{S}_{(2l,-l)}(\mathcal{K}^{(m)})$:=the space of adelic cusp forms of weight $(2l, -l)$ for $\mathcal{K}^{(m)}$

Rem. $\sharp : \bigoplus_{i=1}^{h} S_{(2l,-l)}(\mathit{\Gamma}_i^{(m)}) \cong \mathcal{S}_{(2l,-l)}(\mathcal{K}^{(m)})$

$F = (F_1, ..., F_h)^\sharp \in \mathcal{S}_{(2l,-l)}(\mathcal{K}^{(m)})$
$\langle F, F\rangle := h^{-1} \sum_{i=1}^{h} \langle F_i, F_i\rangle$

R : a commutative ring with an involution
$\mathrm{Her}_m(R)$: the set of Hermitian matrices of degree m with entries in R
$\mathrm{Her}_m^*(\mathcal{O}_p) := \{A = (a_{ij}) \in \mathrm{Her}_m(K_p) \mid a_{ij} \in \sqrt{-D}^{-1}\mathcal{O}_p, a_{ii} \in \mathbf{Z}_p\}$
(the set of semi-integral Hermitian matrices of degree m over $\mathcal{O}_p$)

$T \in \mathrm{Her}_m^*(\mathcal{O}_p),\ \det T \neq 0$

$$b_p(T,s) := \sum_{R \in \mathrm{Her}_m(K_p)/\mathrm{Her}_m(\mathcal{O}_p)} \exp(2\pi\sqrt{-1}\mathrm{tr}(TR))\mu_p(R)^{-s}$$

$\mu_p(R) := [R\mathcal{O}_p^m + \mathcal{O}_p^m : \mathcal{O}_p^m]$

Rem. (G. Shimura) There exists a polynomial $F_p(T,X) \in \mathbf{Z}[X]$ s.t.

$$b_p(T,s) = F_p(T,p^{-s}) \prod_{i=1}^{[(m+1)/2]} (1-p^{2i-s}) \prod_{i=1}^{[m/2]} (1-\xi_p p^{2i-1-s})$$

$$\xi_p := \begin{cases} 1 & \text{if } K_p \cong \mathbf{Q}_p \oplus \mathbf{Q}_p \\ -1 & \text{if } K_p/\mathbf{Q}_p \text{ is unramified quadratic} \\ 0 & \text{if } K_p/\mathbf{Q}_p \text{ is ramified quadratic} \end{cases}$$

$\tilde{F}_p(T,X) := X^{ord_p(\gamma(T))} F_p(T, p^{-m}X^{-2})$

$$R_{k/\mathbf{Q}}(GL_m)_{\mathbf{A}} = \bigsqcup_{i=1}^{h} GL_m(K)t_i \prod_p GL_m(\mathcal{O}_p)GL_m(\mathbf{C}) \quad (t_1 = 1_m)$$

$$t_i = (t_{i,p}) \in R_{K/\mathbf{Q}}(GL_m)_{\mathbf{h}}$$

First let $m = 2n$ and

$$f(z) = \sum_{N=1}^{\infty} c_f(N) \exp(2\pi\sqrt{-1}Nz) : \text{a primitive form in } S_{2k+1}(\Gamma_0(D), \chi)$$

$$p \nmid D, \ \ \alpha_p \in \mathbf{C} \text{ s.t. } \alpha_p + \chi(p)\alpha_p^{-1} = p^{-k}c_f(p), \ \ p \mid D, \ \ \alpha_p := p^{-k}c_f(p)$$

$$T \in \mathrm{Her}_m(K)^+$$

$$\gamma(T) := (-D)^{[m/2]} \det T$$

$$c_{I_{m,i}(f)}(T) := |\gamma(T)|^k \prod_p |\det(t_{i,p})\overline{\det(t_{i,p})}|_p^n \tilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p),$$

$$I_{m,i}(f)(Z) := \sum_{T \in \mathrm{Her}_m(K)^+} c_{I_{m.i}(f)}(T) \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)) \quad (Z \in \mathfrak{H}_m)$$

Next let $m=2n+1$ and

$f(z)$: a primitive form in $S_{2k}(SL_2(\mathbf{Z}))$

In this case we can also define a Fourier series $I_{m,i}(f)$ on $\mathfrak{H}_m$ for f similarly to above.

$$I_{m,i}(f)(Z) := \sum_{T \in \mathrm{Her}_m(K)^+} c_{I_{m,i}(f)}(T) \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)) \quad (Z \in \mathfrak{H}_m)$$

k : a non-negative integer, $m=2n$ or $2n+1$

$$k' = \begin{cases} 2k+1 & \text{if } m=2n, \\ 2k & \text{if } m=2n+1, \end{cases} \quad S^{(m,k')} := \begin{cases} S_{2k+1}(\Gamma_0(D),\chi) & \text{if } m=2n, \\ S_{2k}(SL_2(\mathbf{Z})) & \text{if } m=2n+1. \end{cases}$$

Thm. 2.1. (T. Ikeda 2008) *Let* $m = 2n$ *or* $2n + 1$. *Let* f *be a primitive form in* $S^{(m,k')}$. *Then* $I_{m,i}(f)$ *belongs to* $S_{2k+2n,-k-n}(\Gamma_i^{(m)})$.
(cf. H. Kojima, V. Gritsenko, A. Krieg, T. Sugano).

$Lift^{(m)}(f) := (I_m(f)_1, \cdots, I_m(f)_h)^\sharp$ (the adelic Hermitian Ikeda lift of f)

Thm. 2.2. *(1) Assume that* f *does not come from a Hecke character of* K *if* $m \equiv 2 \bmod 4$. *Then* $Lift^{(m)}(f)$ *is a Hecke eigenform in* $\mathcal{S}_{2k+2n,-k-n}(\mathcal{K}^{(m)})$ *whose standard* L*-function coincides with*

$$\prod_{i=1}^{m} L(s+k+n-i+1/2, f)L(s+k+n-i+1/2, f, \chi).$$

(2) Assume that f *comes from a Hecke character of* K *and that* $m \equiv 2 \bmod 4$. *Then* $Lift^{(m)}(f) = 0$.

3. The period of the Hermitian Ikeda lift

$\Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$

$$\mathbf{L}(j,\chi^j) = \begin{cases} \Gamma_{\mathbf{C}}(2i+1)L(2i+1,\chi)D^{2i+1/2} & j = 2i+1 \\ \Gamma_{\mathbf{C}}(2i)\zeta(2i) & j = 2i \end{cases}$$

$f(z)$: a primitive form in $S_{2k+1}(\Gamma_0(D),\chi)$

$L(s,f,\mathrm{Ad},\chi^i)$

$$:= \prod_{p\nmid D}\{(1-\alpha_p^2\chi(p)^{i+1}p^{-s})(1-\alpha_p^{-2}\chi(p)^{i+1}p^{-s})(1-\chi(p)^i p^{-s})\}^{-1}$$

$$\times \begin{cases} \prod_{p\mid D}(1-p^{-s})^{-1} & \text{if } i \text{ is even} \\ \prod_{p\mid D}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1} & \text{if } i \text{ is odd} \end{cases}$$

$f(z)$: a primitive form in $S_{2k}(SL_2(\mathbf{Z}))$

$$L(s,f,\mathrm{Ad},\chi^i) := \prod_{p}\{(1-\alpha_p^2\chi(p)^i p^{-s})(1-\alpha_p^{-2}\chi(p)^i p^{-s})(1-\chi(p)^i p^{-s})\}^{-1}$$

$f \in S^{(m,k')}$ a primitive form

$$\mathbf{L}(i, f, \mathrm{Ad}, \chi^{i+1}) := \frac{\Gamma_{\mathbf{C}}(i)\Gamma_{\mathbf{C}}(i+k'-1)L(i, f, \mathrm{Ad}, \chi^{i+1})}{\langle f, f\rangle}$$

Rem. $\mathbf{L}(i, f, \mathrm{Ad}, \chi^{i+1}) \in \mathbf{Q}(f)$ if $0 < i \le k'-1$ (Sturm)

Thm. 3.1. *Let* $m = 2n+1$. *Then we have*

$$\frac{\langle I_{m,i}(f), I_{m,i}(f)\rangle}{\langle f, f\rangle^m} = 2^{-4nk-4n^2-6n} \prod_{j=2}^{m} \mathbf{L}(j, f, \mathrm{Ad}, \chi^{j+1})\mathbf{L}(j, \chi^j) D^{2nk+4n^2+n}$$

Rem. If $m = 1$, then $I_{m,1}(f) = f$, and the above equality is trivial.

Q_D:=the set of prime numbers dividing D

$Q \subset Q_D$: $\chi_Q = \prod_{q \in Q} \chi_q \quad \chi'_Q = \prod_{q \in Q_D, q \notin Q} \chi_q$

Rem. $\chi_\emptyset = 1,\ \chi'_\emptyset = \chi, \quad \chi_{Q_D} = \chi,\ \chi'_{Q_D} = 1$

$f(z) = \sum_{N=1}^{\infty} c_f(N) \exp(2\pi\sqrt{-1}Nz) \in S_{2k+1}(\Gamma_0(D), \chi)$: a primitive form

Fact. There exists a unique primitive form

$f_Q(z) = \sum_{N=1}^{\infty} c_{f_Q}(N) \exp(2\pi\sqrt{-1}Nz)$ s.t. $c_{f_Q}(p) = \begin{cases} \chi_Q(p)\overline{c_f(p)} & p \notin Q \\ \chi'_Q(p) c_f(p) & p \in Q \end{cases}$

Rem. $f_\emptyset(z) = f(z),\ f_{Q_D}(z) = \overline{f(-\bar{z})}$

Let $t_i = (t_{i,p})_p \in R_{K/\mathbf{Q}}(GL_m)_{\mathbf{h}}$ be as defined before.
$C(t_i) := \prod_p |N_{K_p/\mathbf{Q}_p}(\det t_{i,p})|_p, \quad \epsilon_i(Q) := \prod_{q\in Q}(-D, C(t_i))_q$
$$\eta_{n,i}(f) := \sum_{\substack{Q \subset Q_D \\ f_Q = f}} \chi_Q((-1)^n)\epsilon_i(Q)$$

Thm. 3.2. *Let $m = 2n$. Then we have*

$$\frac{\langle I_{m,i}(f),\, I_{m,i}(f)\rangle}{\langle f,\, f\rangle^m} = 2^{-4nk+2k+3-4n} \prod_{j=2}^{m} \mathbf{L}(j,\, f,\, \mathrm{Ad}, \chi^{j+1})\mathbf{L}(j, \chi^j)$$

$$\times \eta_{n,i}(f) D^{2nk+4n^2-n} \prod_{q|D} (1 + q^{-1})$$

Rem 1. The case $i = 1$ of Thms. 3.1 and 3.2 was conjectured by Ikeda and was proved by Katsurada.

Rem 2. (Ikeda) Let $m = 2n$. Then we have $\eta_{n,i}(f) = 0$ iff $I_{m,i}(f)$ is identically zero.

$$\tilde{\eta}_n(f) := \begin{cases} 1 + \chi((-1)^n) & \text{if } f_{Q_D} = f \\ 1 & \text{if } f_{Q_D} \neq f \end{cases}$$

Rem. $\sum_{i=1}^{h} \eta_{n,i}(f) = h\tilde{\eta}_n(f)$ (genus theory)

Thm. 3.3. *Assume Theorem 2.1. Then we have*

$$\frac{\langle Lift^{(m)}(f),\, Lift^{(m)}(f)\rangle}{\langle f,\, f\rangle^m} = 2^{\beta_{n,k}} \prod_{i=2}^{m} \mathbf{L}(i,\, f,\, \mathrm{Ad}, \chi^{i+1})\mathbf{L}(i, \chi^i)$$

$$\times \begin{cases} \tilde{\eta}_n(f) D^{2nk+4n^2-n} \prod_{q|D}(1+q^{-1}) & \text{if } m = 2n \\ D^{2nk+4n^2+n} & \text{if } m = 2n+1. \end{cases}$$

Cor. *In addition to the notation and the assumption as Thm. 3.3, assume that* $k' \geq m+1$*. Then* $\dfrac{\langle Lift^{(m)}(f),\, Lift^{(m)}(f)\rangle}{\langle f,\, f\rangle^m}$ *belongs to* $\mathbf{Q}(f)$*.*

Rem. Let $m = 2n$. Then we have $\tilde{\eta}_n(f) = 0$ iff n is odd, and f comes from a Hecke character of K.

4. Outline of the proof of Thms. 3.1 and 3.2.

Main tools:

(1) The residue formula of the Rankin-Selberg series of the $I_{m,i}(f)$

(2) An Explicit formula of the Rankin-Selberg series of $I_{m,i}(f)$

Let $m = 2n$ or $2n+1$, $f \in S^{(m,k')}$.

$$I_{m,i}(f)(Z) := \sum_{B \in \mathrm{Her}_m(K)^+} c_{I_{m,i}(f)}(B) \exp(2\pi\sqrt{-1}\,\mathrm{tr}(BZ)) \quad (Z \in \mathfrak{H}_m).$$

$$U_{m,i} := SL_m(K) \cap SL_m(\mathbf{C})(\prod_p t_{i,p} SL_m(\mathcal{O}_p) t_{i,p}^{-1})$$

$$R(s, I_{m,i}(f)) = C(t_i)^{s-2k-2n} \sum_{B \in \mathrm{Her}_m(K)^+/U_{m,i}} \frac{|c_{I_{m,i}(f)}(B)|^2}{e_i(B)(\det B)^s}$$

$e_i(B) := \#\{g \in U_{m,i} \mid g^* B g = B\}$ $\quad C(t_i) := \prod_p |N_{K_p/\mathbf{Q}_p}(\det t_{i,p})|_p$

Rem. $U_{m,1} = SL_m(\mathcal{O})$, $\quad e_1(B) = \#\{g \in SL_m(\mathcal{O}) \mid g^* B g = B\}$.

$$R(s, I_{m,1}(f)) = \sum_{B \in \mathrm{Her}_m(K)^+/SL_m(\mathcal{O})} \frac{|c_{I_{m,1}(f)}(B)|^2}{e_1(B)(\det B)^s}$$

Prop. 4.1. *$R(s, I_{m,i}(f))$ has a pole at $s = 2k + 2n$ with the residue*

$$\frac{2^{2(k+n)+m-1} \prod_{j=2}^{m} L(j, \chi^{j+1}) \langle I_{m,i}(f), I_{m,i}(f) \rangle}{D^{m(m-1)/2} \prod_{j=1}^{m} (\Gamma_{\mathbf{C}}(j)\Gamma_{\mathbf{C}}(2k+2n-j+1)) \prod_{j=0}^{m-1} L(2m-j, \chi^j)}$$

Prop. 4.2. *Let f be a primitive form in $S^{(m,k')}$. Then*

$$\mathbf{L}(1, f, \mathrm{Ad}, 1) = \begin{cases} 2^{2k+1} \prod_{q|D}(1+q^{-1}) & \text{if } m = 2n \\ 2^{2k} & \text{if } m = 2n+1 \end{cases}.$$

Thm. 4.3. *Let $m = 2n+1$. Then*

$$R(s, I_{2n+1,i}(f)) = D^{ns} 2^{-2n-1} \prod_{j=2}^{2n+1} \mathbf{L}(j, \chi^j) \prod_{j=0}^{2n} L(2s-4k-j+2, \chi^j)^{-1}$$

$$\times \prod_{j=2}^{2n+1} L(s-2k-2n+j, f, \mathrm{Ad}, \chi^{j-1}) L(s-2k-2n+j, \chi^{j-1})$$

$$\times L(s-2k-2n+1, f, \mathrm{Ad}, 1) L(s-2k-2n+1, 1).$$

$Q \subset Q_D$, $f \in \mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$, a primitive form

$$M(s, f, \mathrm{Ad}, \chi_Q) := \prod_{j=1}^{2n} \{ \prod_{p \notin Q} (1 - \alpha_p^2 \chi(p)^j \chi_Q(p) p^{-s+2k+2n-j})$$

$$\times (1 - \alpha_p^{-2} \chi(p)^j \chi_Q(p) p^{-s+2k+2n-j})(1 - \chi^{j-1}(p) \chi_Q(p) p^{-s+2k+2n-j})^2$$

$$\times \prod_{p \in Q} (1 - \alpha_p^2 \chi(p)^j p^{-s+2k+2n-j})(1 - \alpha_p^{-2} \chi(p)^j p^{-s+2k+2n-j})$$

$$\times (1 - \chi^{j-1}(p) p^{-s+2k+2n-j})^2 \}^{-1}.$$

Thm. 4.4. *Let* $m = 2n$.

$$R(s, I_{2n,i}(f)) = D^{ns} 2^{-2n} \prod_{j=2}^{2n} \mathbf{L}(j, \chi^j) \prod_{j=0}^{2n-1} L(2s - 4k - j, \chi^j)^{-1}$$

$$\times \sum_{Q \subset Q_D} \chi_Q((-1)^n) \epsilon_i(Q) M(s - 2k - 2n + j, f, \mathrm{Ad}, \chi_Q).$$

Lem. 4.5. (1) If $f_Q \neq f$, then $M(s, f, \mathrm{Ad}, \chi_Q)$ is holomorphic at $s = 2k + 2n$.
(2) If $f_Q = f$, then

$$M(s, f, \mathrm{Ad}, \chi_Q) = \prod_{j=1}^{2n} L(s - 2k - 2n + j, f, \mathrm{Ad}, \chi^{j-1}) L(s - 2k - 2n + i, \chi^{j-1}).$$

Recall $\eta_{n,i}(f) := \sum_{\substack{Q \subset Q_D \\ f_Q = f}} \chi_Q((-1)^n)\epsilon_i(Q)$. Hence we have

Thm. 4.6. *Let* $m = 2n$.

$$R(s, I_{2n,i}(f)) = D^{ns} 2^{-2n} \prod_{j=2}^{2n} \mathbf{L}(j, \chi^j) \prod_{j=0}^{2n-1} L(2s - 4k - j, \chi^j)^{-1}$$

$$\times \{\eta_{n,i}(f) \prod_{j=1}^{2n} L(s - 2k - 2n + j, f, \mathrm{Ad}, \chi^{j-1}) L(s - 2k - 2n + i, \chi^{j-1})$$

$$\sum_{\substack{Q \subset Q_D \\ f_Q \neq f}} \chi_Q((-1)^n)\epsilon_i(Q) M(s - 2k - 2n + j, f, \mathrm{Ad}, \chi_Q)\}.$$