Some remarks on the trace formula for Jacobi forms of prime power level

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1. **Purpose of this talk.**

1. I tell the deep acknowledgement for being given a opportunity of my talk and many heartful supports on Shanghai life to the all organizers of this conference.

2. We describe the trace formulas for Jacobi forms of prime power level, and give some trace relations between them and the trace formulas for elliptic modular forms. Moreover we consider about the level-index changing operator (the 'swapping' operator) on Jacobi forms in the case of prime odd power level.

2. **General Notation.** We put

$$\mathrm{e}^n(z) = \exp(2\pi\sqrt{-1}nz), \quad \mathrm{e}(z) = \mathrm{e}^1(z) = \exp(2\pi\sqrt{-1}z).$$

Let $S_k(N)$ be the elliptic cusp form space of weight k and level N.

3. **Recall of definition of (scalar-valued) Jacobi forms.**

Let k and m be positive integers and $H(\mathbb{R})$ be the Heisenberg group on $\mathbb{R}^2$. The Jacobi group

$$\mathrm{GL}_2^+(\mathbb{R})^J = \mathrm{GL}_2^+(\mathbb{R}) \ltimes H(\mathbb{R})$$

acts on Hol$(\mathfrak{H} \times \mathbb{C})$ as follows :

$$(1)\ \phi|_{k,m}[g](\tau, z) = (c\tau + d)^{-k} \mathrm{e}^{ml}\left(-\frac{cz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right),$$

$$(2)\ \phi|_{k,m}[((\lambda, \mu), \kappa)](\tau, z) = \mathrm{e}^m\left(\lambda^2\tau + 2\lambda z + \lambda\mu + \kappa\right) \phi(\tau, z + \lambda\tau + \mu)$$

for any $g \in \mathrm{GL}_2^+(\mathbb{R})$ with $\det(g) = l > 0$ and $((\lambda, \mu), \kappa) \in H(\mathbb{R})$.

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a Fuchs group of finite index.

We say a function $\phi(\tau, z)$ of Hol$(\mathfrak{H} \times \mathbb{C})$ is a (holomorphic) Jacobi cusp form of weight k and index m with respect to $\Gamma^J = \Gamma \ltimes H(\mathbb{Z})$ if it satisfies the following three conditions :

(1) $\phi|_{k,m}[g] = \phi$ for any $g \in \Gamma$.

(2) $\phi|_{k,m}[(\lambda,\mu)] = \phi$ for any $(\lambda,\mu) \in \mathbb{Z}^2$.

(3) For any $g \in \mathrm{GL}_2^+(\mathbb{Z})$, $\phi|_{k,m}[g]$ has the Fourier expansion in the following

$$\phi|_{k,m}[g](\tau,z) = \sum_{\substack{(n,r)\in\mathbb{Q}^2 \\ 4mn-r^2>0}} c^g(n,r)\mathrm{e}(n\tau + rz).$$

We denote the space of Jacobi cusp forms of weight k and index m with respect to Γ^J by $J_{k,m}^{\mathrm{cusp}}(\Gamma^J)$. Especially, if $\Gamma^J = \Gamma_0(N)^J$, we denote $J_{k,m}^{\mathrm{cusp}}(\Gamma^J)$ by $J_{k,m}^{\mathrm{cusp}}(N)$.

For a finite union Δ of double Γ^J-cosets in $\mathrm{SL}_2(\mathbb{Q})^J$, we define an operator on $J_{k,m}^{\mathrm{cusp}}(\Gamma)$ as follows:

$$\phi|H_{k,m,\Gamma}(\Delta) = \sum_{\xi\in\Gamma^J\backslash\Delta} \phi|_{k,m}\xi. \quad \text{(This is a finite sum.)}$$

For a positive integer l with $(l, Nm) = 1$, we define the l-th Hecke operator $T(l)$ on $J_{k,m}^{\mathrm{cusp}}(N)$ as follows:

$$\phi|T(l) = l^{k-4} \sum_{\substack{l'|l \\ l/l'=\square}} \frac{l^2}{l'} \phi|H_{k,m,\Gamma_0(N)}\left(\Gamma_0(N)^J \begin{pmatrix} l'^{-1} & 0 \\ 0 & l' \end{pmatrix} \Gamma_0(N)^J\right).$$

Theorem 1 (S (2009,2010)). *Suppose that k is an even natural number with $k \geq 4$, N is a positive odd squarefree integer and l is a natural number with $(l, N) = 1$. Then we have the following*

$$\mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(N)\right) = \mathrm{tr}\left(T(l), S_{2k-2}(N)\right),$$

$$\mathrm{tr}\left(T(l) \circ W_{\mathsf{L}}(n), J_{k,1}^{\mathrm{cusp}}(N)\right) = \mathrm{tr}\left(T(l) \circ W_n, S_{2k-2}(N)\right).$$

Here W_n with $n|N$ is the Atkin-Lehner operator on elliptic cusp forms.

This theorem in the case of $N = 1$ was derived by Skoruppa-Zagier(1989). The proof of this theorem in the case of general squarefree level is given by obtaining generalizations of Skoruppa-Zagier's methods.

4. **Sketch of Proof of Theorem 1.**

We calculate the following two traces in this case explicitly and compare them. (they are very hardwork!)

Trace formula for elliptic modular forms (Eichler,Selberg, Oesterlé,Yamauchi,Hijikata and Zagier)

Let k be a positive integer with $k \geq 2$ and (n_1, n_2) are positive integers pair of relative prime. Then we have

$$\mathrm{tr}\left(T(l) \circ W_{n_1}\ ;\ S_{2k-2}(n_1 n_2)\right)$$

$$= -\frac{1}{2} \sum_{\substack{n' | n_1 \\ n_1/n' = \square}} \mu\left(\sqrt{n_1/n'}\right) \sum_{\substack{s^2 \leq 4ln' \\ \sqrt{n_1 n'} | s}} p_{2k-2}\left(s/\sqrt{n'},\, l\right) \sum_{\substack{t | n_2 \\ n_2/t;\,\text{squarefree}}} H_t(s^2 - 4ln')$$

$$-\frac{1}{2}\delta(n_1 = \square)\varphi\left(\sqrt{n_1}\right) \sum_{\substack{l' | l \\ \sqrt{n_1} | \left(l' + \frac{l}{l'}\right)}} \min\left(l', \frac{l}{l'}\right)^{2k-3} \sum_{\substack{t | n_2 \\ n_2/t;\,\text{squarefree}}} \left(Q(t), \left(l' - \frac{l}{l'}\right)\right)$$

$$+\delta(k = 2)\sigma_1(l).$$

Remark 1. Here we put

$$(1)\quad p_k(s,l)=\begin{cases}\dfrac{\rho^{k-1}-\rho'^{k-1}}{\rho-\rho'}\ (\rho,\rho';\text{the roots of } X^2-sX+l=0) \text{ if } s^2-4l\neq 0,\\ (k-1)\left(\dfrac{s}{2}\right)^{k-2} \text{ if } s^2-4l=0.\end{cases}$$

$$(2)\quad H_n(\Delta)=\begin{cases}a^2b\left(\dfrac{\Delta/a^2b^2}{n/a^2b}\right)H_1(\Delta/a^2b^2) \text{ if } a^2b^2|\Delta,\\ 0 \text{ if otherwise.}\end{cases}$$

where $H_1(0)=-\dfrac{1}{12}$ and $H_1(\Delta)$ for Δ is the number of equivalence classes with respect to $\mathrm{SL}_2(\mathbb{Z})$ of integral,positive definite, binary quadratic forms of discriminant Δ, counting forms equivalent to a multiple of x^2+y^2 (resp. x^2+xy+y^2) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$). $H_1(\Delta)$ is called the Hurwitz-Kronecker class number.

Trace formula for Jacobi forms (Skoruppa-Zagier)

Suppose that $k > 2$, Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, $m \geq 1$ and Δ is a finite union of double Γ^J-cosets in $\mathrm{SL}_2(\mathbb{Q})^J$. Then we have

$$\mathrm{tr}\left(H_{k,m,\Gamma}(\Delta);\ J_{k,m}^{\mathrm{cusp}}(\Gamma)\right) = \sum_{A \in \mathrm{Pr}(\Delta)/\sim_{m,\Gamma}} I_{k,m,\Gamma}(A) g_m(\Delta, A),$$

where $\mathrm{Pr} : \mathrm{SL}_2(\mathbb{Q})^J \to \mathrm{SL}_2(\mathbb{Q})$ denotes the canonical projection and $\sim_{m,\Gamma}$ is the equivalence relation defined by $A \sim_{m,\Gamma} B$ if and only if

$$\begin{aligned} &(1)\ A \text{ and } B \text{ are } \Gamma-\text{conjugate, or} \\ &(2)\ A \text{ and } B \text{ are parabolic and } GA \text{ is } \Gamma-\text{conjugate to } B \\ ,\quad &\quad\ \ \text{for some } G \in \Gamma_A \cap \Gamma(4m).\ (\Gamma_A = \{G \in \Gamma \,|\, GAG^{-1} = A\}) \end{aligned}$$

Here $I_{k,m,\Gamma}(A)$ is the contribution term of modular element A, and $g_m(\Delta, A)$ is certain limit of exponential quadraric sum over lattice (This term changes class numbers which is represented by H_t in many cases).

Corollary 1 (S (2009,2010)). *From the above Theorem 1, we obtain the following*

$$\mathrm{tr}(T(l), J_{k,1}^{\text{cusp,new},+\cdots+}(N)) = \mathrm{tr}(T(l), S_{2k-2}^{\text{new},+\cdots+}(N)),$$

where $J_{k,1}^{\text{cusp,new},+\cdots+}(N)$ *denotes the space of all forms* $\phi \in J_{k,1}^{\text{cusp,new}}(N)$ *satisfying* $\phi|W_{\mathsf{L}}(n) = \phi$ $(n|N)$, *and* $S_{2k-2}^{\text{new},+\cdots+}(N)$ *denotes the space of all forms* $f \in S_{2k-2}^{\text{new}}(N)$ *with* $f|W_n = f$ $(n|N)$.

Remark 2. For $(Q, N/Q) = 1$, $W_{\mathsf{L}}(Q)$ is the Atkin-Lehner operator with respect to level on $J_{k,1}(N)$ as follows:

$$\phi|W_{\mathsf{L}}(Q) = \frac{1}{Q^2}\phi|H_{k,1,\Gamma_0(N)}\left(\frac{1}{Q}\Gamma_0(N)^J\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\begin{pmatrix}Qa & b\\ Nc & Qd\end{pmatrix}\Gamma_0(N)^J\right),$$

where a, b, c and d are integers with $Q^2ad - Nbc = Q$.

Corollary 2 (S (2009,2010)). *Notation and assumptions are the same as above. Then, from Corollary 1 and the result of Skoruppa-Zagier, we get the following*

$$\mathrm{tr}(T(l), J_{k,1}^{\text{cusp,new},+\cdots+}(N)) = \mathrm{tr}(T(l), J_{k,N}^{\text{cusp,new},+\cdots+}(1)).$$

Here $J_{k,N}^{\text{cusp,new},+\cdots+}(1)$ *is the subspace of* $J_{k,N}^{\text{cusp,new}}(1)$ *consisting of all* ϕ *which satisfies* $\phi|W_{\mathrm{I}}(n) = \phi$ $(n|N)$.

Remark 3. For $n||m$, $W_{\mathrm{I}}(n)$ is the Atkin-Lehner operator with respect to index on $J_{k,m}(1)$

$$\begin{aligned}\phi|W_{\mathrm{I}}(n) &= \frac{1}{n^2}\phi|H_{k,m,\Gamma_0(N)}\left(\Gamma_0(N) \ltimes \left(\frac{1}{n}\mathbb{Z}^2\right)\cdot \mu_n\right) \\ &= \sum_{D \leqq 0}\left(\sum_{\substack{r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} c_\phi(D,\lambda_n r)\right) \mathrm{e}\left(\frac{r^2-D}{4m}\tau + rz\right),\end{aligned}$$

where λ_n is the modulo $2m$ uniquely determined integer which satisfies $\lambda_n \equiv -1 \pmod{2n}$ and $\lambda_n \equiv +1 \pmod{2m/n}$.

Theorem 2 (S (2009,2010)). *The notation and assumption is the same as above. For a squarefree odd N, we obtain the Lifting map from $J_{k,1}^{\mathrm{cusp,new},+\cdots+}(N))$ to $J_{k,N}^{\mathrm{cusp,new},+\cdots+}(1)$ as follows:*

$$\sum_{\substack{0>D,r\in\mathbf{Z} \\ D\equiv r^2 \,(\mathrm{mod} 4)}} c_\phi(D;r)\mathrm{e}\left(\frac{r^2-D}{4}\tau+rz\right) \to \sum_{\substack{0>D,r\in\mathbf{Z} \\ D\equiv r^2 \,(\mathrm{mod} 4N)}} c_\phi(D;r)\mathrm{e}\left(\frac{r^2-D}{4N}\tau+rz\right).$$

Remark 4. In the case of N being not squarefree, the coincidence of Hecke Traces does not work in the same form as Theorem 1. Therefore we have not the level-index changing phenomenon on these spaces.

We construct the same isomorphism as them in the case of prime power level by restricting the special subspace.

Theorem 3 (S (2013)). *$k \in 2\mathbb{N}_{\geq 2}$, p: odd prime and $(l,p)=1$.*

(1) $$\mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(p^2)\right)_{\mathrm{elliptic}} - \mathrm{tr}\left(T(l), S_{2k-2}(p^2)\right)_{\mathrm{elliptic}}$$
$$= -\frac{1}{2}\sum_{\substack{s^2<4l\\ s\neq 0}} p_{2k-2}(s,l)H_1(s^2-4l)\left(\frac{\Delta_{(s,l)}/p^2}{p}\right),$$

(2) $$\mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(p^4)\right)_{\mathrm{elliptic}} - \mathrm{tr}\left(T(l), S_{2k-2}(p^4)\right)_{\mathrm{elliptic}}$$
$$= -\frac{1}{2}\sum_{\substack{s^2<4l\\ s\neq 0}} p_{2k-2}(s,l)H_1(s^2-4l)$$
$$\times\left(\left(1+\left(\frac{\Delta_{(s,l)}/p^2}{p}\right)\right)\left(\frac{\Delta_{(s,l)}/p^2}{p}\right)+p\left(\frac{\Delta_{(s,l)}/p^4}{p}\right)\right),$$

(3) $$\mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(p^6)\right)_{\mathrm{elliptic}} - \mathrm{tr}\left(T(l), S_{2k-2}(p^6)\right)_{\mathrm{elliptic}}$$
$$= -\frac{1}{2}\sum_{\substack{s^2<4l\\ s\neq 0}} p_{2k-2}(s,l)H_1(s^2-4l)$$
$$\times\left(\left(1+\left(\frac{\Delta_{(s,l)}/p^2}{p}\right)\right)\left(\frac{\Delta_{(s,l)}/p^2}{p}\right)+p\left(1+\left(\frac{\Delta_{(s,l)}/p^4}{p}\right)\right)+p^2\left(\frac{\Delta_{(s,l)}/p^6}{p}\right)\right).$$

Here $\Delta_{(s,l)}=s^2-4l$ and $\left(\frac{a}{b}\right)=0$ if $a\notin\mathbb{Z}$ or $(a,b)>1$.

Theorem 4 (S (2013)).

$$
\begin{aligned}
& \operatorname{tr}\left(T(l), J_{k,1}^{\text{cusp}}(p^{2m+1})\right)_{\text{elliptic}} - \operatorname{tr}\left(T(l), S_{2k-2}(p^{2m+1})\right)_{\text{elliptic}} \\
= \; & \operatorname{tr}\left(T(l), J_{k,1}^{\text{cusp}}(p^{2m})\right)_{\text{elliptic}} - \operatorname{tr}\left(T(l), S_{2k-2}(p^{2m})\right)_{\text{elliptic}} \\
& -\frac{1}{2} \cdot p^{m-1} \sum_{\substack{s^2<4l \\ s\neq 0,\, p^{2m} \| (s^2-4l)}} p_{2k-2}(s,l) H_1(s^2-4l).
\end{aligned}
$$

That is, almost all terms in the trace relation of odd prime power level p^{2m+1} coincides with them in the trace relation of even prime power level p^{2m}.

Remark 5. We have already got the another terms of trace relations, that is, hyperbolic contributions, parabolic contributions and scalar contributions.

Remark 6. The structure of the trace formula of odd prime power level p^{2m} is very complicated. Therefore we have not obtained some relations of the p^{2m} case and the p^{2m-1} case yet (Note that all terms divisible by even power in this trace relation has left!). This situation is a similar case to the trace formula for modular forms of half-integral weight. For example, we have

$$\begin{aligned}
\mathrm{tr}\left(T(l^2), S_{k+1/2}^{K,\mathrm{new}}(4p^{2m+1})^{\pm}\right) &= \mathrm{tr}\left(T(l), S_{2k}^{\mathrm{new},\pm\left(\frac{-1}{p}\right)^k}(p^{2m+1})\right), \\
\mathrm{tr}\left(T(l^2), S_{k+1/2}^{K,\mathrm{new}}(4p^{2m})^{\pm}\right) &= \mathrm{tr}\left(T(l), S_{2k}^{\mathrm{new},+}(p^{2m})\right)
\end{aligned}$$

for $m \geq 2$, where $S_{k+1/2}^{K,\mathrm{new}}(4p^l)^{\pm}$ is the $\pm$-eigen subspace of the Kohnen newform space $S_{k+1/2}^{K,\mathrm{new}}(4p^l)$ with respect to the twisting operator. Therefore $S_{k+1/2}^{K,\mathrm{new}}(4p^{2m+1})$ holds the multiplicity one Theorem, but $S_{k+1/2}^{K,\mathrm{new}}(4p^{2m})$ does not hold. This result indecates that $S_{k+1/2}^{K,\mathrm{new}}(4p^{2m})$ has more difficult strucuture than $S_{k+1/2}^{K,\mathrm{new}}(4p^{2m+1})$.

Corollary 3 (S (2013)). *The assumption is the same as above. Then we have*

$$\begin{aligned} & \mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(p^{2m+1})\right) - \mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp}}(p^{2m})\right) \\ = \; & \mathrm{tr}\left(T(l), S_{2k-2}(p^{2m+1})\right) - \mathrm{tr}\left(T(l), S_{2k-2}(p^{2m})\right) \\ & + p^{m-1}\mathrm{tr}\left(T(l) \circ W_{p^{2m}}, S_{2k-2}(p^{2m})\right). \end{aligned}$$

In other word, we obtain the following

$$\mathrm{tr}\left(T(l), J_{k,1}^{\mathrm{cusp},*}(p^{2m+1})\right) = \mathrm{tr}\left(T(l), S_{2k-2}^{*}(p^{2m+1})\right),$$

where $J_{k,1}^{\mathrm{cusp},*}(p^{2m+1})$ *is the orthogonal complement subspace of* $J_{k,1}^{\mathrm{cusp,old}}(p^{2m+1})$ *in* $J_{k,1}^{\mathrm{cusp}}(p^{2m+1})$*, and* $S_{2k-2}^{*}(p^{2m+1})$ *is the orthogonal complement subspace of* $S_{2k-2}^{\mathrm{old}}(p^{2m+1})$ *in* $S_{2k-2}(p^{2m+1})$*.*

5. Construction of the Lifting map in the case of level p^{2m+1}.

Theorem 5 (S (2013)). *The notation and assumption is the same as above. For $N = p^{2m+1}$, we obtain the Lifting map from $J_{k,1}^{\mathrm{cusp},*,+}(N)$ to $J_{k,N}^{\mathrm{cusp},*,+}(1)$ as follows:*

$$\sum_{\substack{0>D,r\in\mathbf{Z}\\ D\equiv r^2\,(\mathrm{mod}4)}} c_\phi(D;r)\mathrm{e}\left(\frac{r^2-D}{4}\tau+rz\right) \to \sum_{\substack{0>D,r\in\mathbf{Z}\\ D\equiv r^2\,(\mathrm{mod}4N)}} c_\phi(D;r)\mathrm{e}\left(\frac{r^2-D}{4N}\tau+rz\right).$$

Remark 7. This lifting map is reconstructed in the framework of the Weil representation (joint work with Prof. N.Skoruppa and Prof. H.Aoki) .

Vielen Dank!