# *Hecke Operators on Siegel Eisenstein Series*

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While "classical" Eisenstein series are well-understood, there remain many unanswered questions regarding Siegel Eisenstein series. Perhaps most distressing is that we do not formulas for the Fourier coefficients for a basis of Siegel Eisenstein series of level $\mathcal{N} > 1$, with arbitrary degree and character.

In this talk I will describe how to evaluate the action of the Hecke operators on Siegel Eisenstein series of arbitrary degree, level, and character, and discuss how these results can be used to find the Fourier coefficients of at least some of the Siegel Eisenstein series.

**Recall:**

$$Sp_n(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{2n}(\mathbb{Z}) : \ A\,{}^tB,\ C\,{}^tD \text{ symmetric}, \right.$$
$$\left. A\,{}^tD - B\,{}^tC = I \ \right\}.$$

Subgroups of importance to us include

$$\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \right\},$$

$$\Gamma_\infty^+ = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : \ \det A = 1 \right\},$$

$$\Gamma(\mathcal{N}) = \{\gamma \in Sp_n(\mathbb{Z}) : \ \gamma \equiv I \ (\mathcal{N}) \ \},$$

$$\Gamma_0(\mathcal{N}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : \ C \equiv 0 \ (\mathcal{N}) \right\};$$

here $\mathcal{N} \in \mathbb{Z}_+$.

It is well-known that for

$$\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix}, \gamma' = \begin{pmatrix} * & * \\ M' & N' \end{pmatrix} \in Sp_n(\mathbb{Z}),$$

we have $\gamma' \in \Gamma_\infty^+ \gamma$ if and only if $(M' \ N') \in SL_n(\mathbb{Z})(M \ N)$ .

Suppose

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z});$$

then $(M \ N)$ is a coprime symmetric pair, meaning that $M, N$ are integral, $M\,{}^tN$ is symmetric, and for every prime $p$, $\mathrm{rank}_p(M \ N) = n$, where $\mathrm{rank}_p$ denotes the rank over $\mathbb{Z}/p\mathbb{Z}$. On the other hand, given any coprime symmetric pair of $n \times n$ matrices $(M \ N)$, there exists some $\begin{pmatrix} K & L \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$.

**Defining Siegel Eisenstein series:**

Take $n, k, \mathcal{N} \in \mathbb{Z}_+$, $\chi$ a character modulo $\mathcal{N}$. For

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}),$$

$$\tau \in \mathcal{H}_n = \{X + iY : \ X, Y \in \mathbb{R}^{n,n}_{\text{sym}}, \ \ Y > 0 \ \},$$

set

$$1(\tau)| \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(C\tau + D)^{-k}.$$

For $\gamma \in Sp_n(\mathbb{Z})$, we *want* to define

$$\mathbb{E}_\gamma(\tau) = \sum \overline{\chi}(\delta)\, 1(\tau)|\gamma\delta$$

where $\delta \in \Gamma_0(\mathcal{N})$ varies so that $\Gamma_\infty^+ \gamma \Gamma_0(\mathcal{N}) = \cup_\delta \Gamma_\infty^+ \gamma\delta$ (disjoint).

For $\gamma \in \Gamma_0(\mathcal{N})$, $\delta^* \in \Gamma_\infty^+$, we have $1(\tau)|\delta^*\gamma = 1(\tau)|\gamma$.
However, this sum still may not be well-defined, so instead we over-sum so that $\mathbb{E}_\gamma$ is as above when the above sum is well-defined, and $\mathbb{E}_\gamma = 0$ otherwise.

Provided $k > n + 1$, we get analytic functions.

With $\{\gamma_\sigma\}$ a set of representatives for $\Gamma_\infty\backslash Sp_n(\mathbb{Z})/\Gamma_0(\mathcal{N})$, the nonzero $\mathbb{E}_{\gamma_\sigma}$ are linearly independent, so they form a basis for the space of Siegel Eisenstein series.

We can (and do) choose these $\gamma_\sigma$ of the form $\begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix}$.

We often write $\mathbb{E}_\sigma$ for $\mathbb{E}_{\gamma_\sigma}$.

**Note:**

$$\mathbb{E}_\sigma(\tau) = \sum_{(M\ \ N)} \overline{\chi}(M,N)\ \det(M\tau + N)^{-k}$$

where $SL_n(\mathbb{Z})(M\ \ N)$ varies over $SL_n(\mathbb{Z})(M_\sigma\ \ I)\Gamma_0(\mathcal{N})$, and $\chi(M,N) = \chi(\delta)$ where $SL_n(\mathbb{Z})(M\ \ N) = SL_n(\mathbb{Z})(M_\sigma\ \ I)\delta$.

When $\mathcal{N}$ is square-free, each $\gamma_\sigma$ is associated to a "multiplicative partition" $(\mathcal{N}_0, \ldots, \mathcal{N}_n)$ of $\mathcal{N}$ (so $\mathcal{N}_0 \cdots \mathcal{N}_n = \mathcal{N}$), and we can take $M_\sigma$ diagonal with

$$M_\sigma \equiv 0 \ (\mathcal{N}_0), \ \ M_\sigma \equiv \begin{pmatrix} I_d & \\ & 0_{n-d} \end{pmatrix} \ (\mathcal{N}_d) \text{ for } 0 < d \leq n.$$

In this case $\mathbb{E}_\sigma \neq 0$ if and only if $\chi_q^2 = 1$ for all primes $q | \mathcal{N}$ so that $0 < \operatorname{rank}_q M_\sigma < n$.

*Theorem*

*Suppose* $\mathcal{N}$ *is square-free. Fix a prime* $q|\mathcal{N}$ *and a multiplicative partition* $\sigma' = (\mathcal{N}_0', \ldots, \mathcal{N}_n')$ *of* $\mathcal{N}/q$*; for* $0 \leq d \leq n$*, let* $\sigma_d = (\mathcal{N}_0, \ldots, \mathcal{N}_n)$ *where*

$$\mathcal{N}_i = \begin{cases} \mathcal{N}_i' & \text{if } i \neq d, \\ q\mathcal{N}_d' & \text{if } i = d. \end{cases}$$

*Then when* $\mathbb{E}_{\sigma_d} \neq 0$*, we have*

$$\mathbb{E}_{\sigma_d} | T(q) = q^{kd-d(d+1)/2} \overline{\chi}_{\mathcal{N}/q} \left(q X_d^{-1} M_{\sigma_d}, X_d^{-1}\right) \cdot \sum_{t=0}^{n-d} q^{-dt-t(t+1)/2} \beta_q(d+t, t) \, \mathrm{sym}_q^{\chi}(t) \, \mathbb{E}_{\sigma_{d+t}}.$$

Here

$$X_d = \begin{pmatrix} qI_d & \\ & I_{n-d} \end{pmatrix}, \; \mathrm{sym}_q^{\chi}(t) = \sum_U \chi_q(\det U),$$

where $U$ runs over $\mathbb{Z}_{\mathrm{sym}}^{t,t}$ modulo $q$, and

$$\beta_q(d+t,t) = \prod_{i=0}^{t-1} \frac{(q^{d+t-i}-1)}{(q^{t-i}-1)}$$

(the number of $t$-dimensional subspaces of a $d+t$-dimensional space over $\mathbb{Z}/q\mathbb{Z}$.)

**Idea of proof:**

$$\mathbb{E}_{\sigma_d}(\tau)|T(q) = q^{-n(n+1)/2} \sum_{M,N,Y} \overline{\chi}(M,N) \det(M\tau/q + MY/q + N)^{-k}$$

where $SL_n(\mathbb{Z})(M\ \ N)$ varies over $SL_n(\mathbb{Z})(M_{\sigma_d}\ \ I)\Gamma_0(\mathcal{N})$ and $Y = \mathbb{Z}^{n,n}_{\mathrm{sym}}$, varying modulo $q$.

For $(M\ \ N) \in SL_n(\mathbb{Z})(M_{\sigma_d}\ \ I)\Gamma_0(\mathcal{N})$, we have $\mathrm{rank}_q\, M = d$.
Adjusting the representative $(M\ \ N)$ using $SL_n(\mathbb{Z})$, we can assume $q$ divides the lower $n-d$ rows of $M$.
Then

$$(M'\ \ N') = \begin{pmatrix} qI_d & \\ & I \end{pmatrix} (M/q\ \ \ MY/q + N)$$

is a coprime symmetric pair with $\mathrm{rank}_q\, M' = d' \geq d$.

Also, $\det(M\tau/q + MY/q + N)^{-k} = q^{kd}\det(M'\tau + N')^{-k}$.

Reversing, given $(M' \ N') \in SL_n(\mathbb{Z})(M_{\sigma_{d'}} \ I)\Gamma_0(\mathcal{N})$, we need to **count** (weighted by the character $\chi$) the equivalence classes

$$SL_n(\mathbb{Z})(M \ N) \in SL_n(\mathbb{Z})(M_{\sigma_d} \ I)\Gamma_0(\mathcal{N})$$

so that

$$\begin{pmatrix} qI_d & \\ & I \end{pmatrix} (M/q \quad MY/q + N) \in SL_n(\mathbb{Z})(M' \ N').$$

So

$$\mathbb{E}_{\sigma_d}(\tau)|T(q) = q^{kd-n(n+1)/2} \sum_{(M' \ N')} c_d(M', N') \det(M'\tau + N')^{-k},$$

for some $c_d(M', N')$.

Also, $\mathbb{E}_{\sigma_d}|T(q)$ is a Siegel Eisenstein series, so

$$\mathbb{E}_{\sigma_d}|T(q) = q^{kd-n(n+1)/2} \sum_{d' \geq d} c_d(M_{\sigma_{d'}}, I)\mathbb{E}_{\sigma_{d'}}.$$

So we only need to compute these $c_d(M_{\sigma_{d'}}, I)$, and then we get the theorem. □

So

$$\mathbb{E}_{\sigma_d}|T(q) = \sum_{d' \geq d} c_d(M_{\sigma_{d'}}, I)\mathbb{E}_{\sigma_{d'}}$$

and for $d \neq d'$, we have

$$c_d(M_{\sigma_d}, I) \neq c_{d'}(M_{\sigma_{d'}}, I).$$

Therefore we can diagonalise

$$\text{span}\{\mathbb{E}_{\sigma_0}, \dots, \mathbb{E}_{\sigma_n}\}.$$

This gives us the following result.

**Dfn.** With $\sigma, \alpha$ multiplicative partitions of $\mathcal{N}$, $Q|\mathcal{N}$, write
$\sigma < \alpha \ (Q)$ if $\text{rank}_q M_\sigma < \text{rank}_q M_\alpha$ for all primes $q|Q$, and
$\sigma = \alpha \ (Q)$ if $\text{rank}_q M_\sigma = \text{rank}_q M_\alpha$ for all primes $q|Q$,

*Corollary*

*Suppose $\mathcal{N}$ is square-free, and $\sigma$ is a multiplicative partition of $\mathcal{N}$ so that $\mathbb{E}_\sigma \neq 0$, and let $q$ be a prime dividing $\mathcal{N}$. There are $a_{\sigma,\alpha}(q) \in \mathbb{C}$ so that*

$$\mathbb{E}_\sigma + \sum_{\substack{\alpha = \sigma \ (\mathcal{N}/q) \\ \alpha > \sigma \ (q)}} a_{\sigma,\alpha}(q)\, \mathbb{E}_\alpha$$

*is an eigenform for $T(q)$, with eigenvalue*

$$\lambda_\sigma(q) = q^{kd - d(d+1)/2} \chi_{\mathcal{N}/q}(\overline{q} X_d M_\sigma, X_d).$$

*Corollary*

*Say $\mathcal{N}$ is square-free, and $\mathbb{E}_\sigma \neq 0$. For $Q|\mathcal{N}$ and $\alpha \geq \sigma\ (Q)$, set*

$$a_{\sigma,\alpha}(Q) = \prod_{q|Q} a_{\sigma,\alpha}(q).$$

*With*

$$\widetilde{\mathbb{E}}_\sigma = \sum_{\alpha \geq \sigma\ (\mathcal{N})} a_{\sigma,\alpha}(\mathcal{N})\mathbb{E}_\alpha,$$

*$\widetilde{\mathbb{E}}_\sigma$ is an eigenform for all $T(q)$, $q|\mathcal{N}$ with eigenvalue $\lambda_\sigma(q)$. If $\sigma \neq \rho\ (\mathcal{N})$, then $\exists\ q|\mathcal{N}$ so that $\lambda_\sigma(q) \neq \lambda_\rho(q)$ (i.e. we have "multiplicity-one").*

**Idea of proof:**

$$\widetilde{\mathbb{E}}_\sigma = \sum_{\alpha \geq \sigma \ (\mathcal{N})} a_{\sigma,\alpha}(\mathcal{N}) \mathbb{E}_\alpha$$

$$= \sum_{\substack{\beta = \sigma \ (q) \\ \beta \geq \sigma \ (\mathcal{N}/q)}} a_{\sigma,\beta}(\mathcal{N}/q) \sum_{\substack{\alpha \geq \beta \ (q) \\ \alpha = \beta \ (\mathcal{N}/q)}} a_{\beta,\alpha}(q) \mathbb{E}_\alpha .$$

So

$$\widetilde{\mathbb{E}}_\sigma | T(q) = \sum_\beta a_{\sigma,\beta}(\mathcal{N}/q) \lambda_\beta(q) \sum_\alpha a_{\beta,\alpha}(q) \mathbb{E}_\alpha .$$

We argue that $\lambda_\beta(q) = \lambda_\sigma(q)$ when $a_{\sigma,\beta}(\mathcal{N}/q) \neq 0$. $\square$

Similar to the first theorem, one proves the following.

*Theorem*

*Assume $\mathcal{N}$ is square-free, and fix a prime $q|\mathcal{N}$. With notation as above, suppose $\mathbb{E}_{\sigma_d} \neq 0$. Then for $0 \leq j \leq n$,*

$$\mathbb{E}_{\sigma_d} | T_j(q^2) = \sum_{t=0}^{n-d} A_j(d,t)\mathbb{E}_{\sigma_{d+t}} \text{ where}$$

$$\begin{aligned} A_j(d,t) = q^{(j-t)d-t(t+1)/2}&\boldsymbol{\beta}_q(d+t,t) \\ &\cdot \sum_{d_1=0}^{j}\sum_{d_5=0}^{j-d_1}\sum_{d_8=0}^{d_5} q^* \,\overline{\chi}_{\mathcal{N}/q}(*,*) \\ &\cdot \boldsymbol{\beta}_q(d,d_1)\boldsymbol{\beta}_q(t,d_5)\boldsymbol{\beta}_q(n-d-t,d_1+n-d-j-d_8) \\ &\cdot \boldsymbol{\beta}_q(t-d_5,d_8)\,\mathrm{sym}_q^{\chi}(t-d_5-d_8)\,\mathrm{sym}_q^{\chi}(d_5,d_8). \end{aligned}$$

*YIKES!*

(Here $\mathrm{sym}_q^{\chi}(b, c)$ is the sum of $\chi_q(\det U)$ where $U$ varies over symmetric matrices modulo $q$, of size $(b+c) \times (b+c)$ and whose lower $c \times c$ block is $0$ modulo $q$.)

*Corollary*

*With* $\mathbb{E}_\sigma \neq 0$, *we have*

$$\widetilde{\mathbb{E}}_\sigma | T_j(q^2) = \lambda_{\sigma,j}(q^2) \widetilde{\mathbb{E}}_\sigma$$

*where*

$$\lambda_{\sigma,j}(q^2) = q^{jd} \sum_{\ell=0}^{j} q^{\ell(2k-2d-j+\ell-1)} \chi_{\mathcal{N}_0}(q^{2\ell}) \overline{\chi}_{\mathcal{N}_n}(q^{2(j-\ell)}) \cdot \boldsymbol{\beta}_q(d,\ell) \boldsymbol{\beta}_q(n-d, j-\ell).$$

From earlier work with J.L. Hafner, we have formulas for the action of the Hecke operators on Fourier coefficients. So when $\chi = 1$, using the fact that

$$\sum_{\sigma} \mathbb{E}_{\sigma}$$

is the Eisenstein series of level 1, with courage, one can use known Fourier expansions for the level 1 Eisenstein series and the above Theorems, and hope to obtain Fourier expansions for all $\mathbb{E}_{\sigma}$.

(See work of Martin Dickson for a solution to this problem in degree 2.)

Now let $\mathcal{N}$ be arbitrary.

**Dfn.** For $v, w \in \mathbb{Z}$ with $(vw, \mathcal{N}) = 1$, $\gamma = \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \in Sp_n(\mathbb{Z})$, set

$$(v, w) \cdot M = v \begin{pmatrix} w & \\ & I \end{pmatrix} M \begin{pmatrix} w & \\ & I \end{pmatrix},$$

$$(v, w) \cdot \gamma = \begin{pmatrix} I & 0 \\ (v, w) \cdot M & I \end{pmatrix},$$

$$(v, w) \cdot \mathbb{E}_\gamma = \mathbb{E}_{(v,w)\cdot\gamma}.$$

*Proposition*

*With $\mathcal{U}_\mathcal{N} = (\mathbb{Z}/\mathcal{N}\mathbb{Z})^\times$, we have a group action of $\mathcal{U}_\mathcal{N} \times \mathcal{U}_\mathcal{N}$ on*

$$\left\{ \mathbb{E}_\gamma : \ \gamma = \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \in Sp_n(\mathbb{Z}) \right\}.$$

Fix a set of representatives $\{\gamma_\sigma\}$ for $\Gamma_\infty \backslash Sp_n(\mathbb{Z})/\Gamma_0(\mathcal{N})$ so that

$$\gamma_\sigma = \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix}.$$

To ease notation, write $\mathbb{E}_\sigma$ for $\mathbb{E}_{\gamma_\sigma}$, and $\mathbb{E}_{(v,w)\cdot\sigma}$ for $\mathbb{E}_{(v,w)\cdot\gamma_\sigma}$.

*Theorem*

*Suppose* $\mathbb{E}_\sigma \neq 0$; *fix a prime* $p \nmid \mathcal{N}$. *Take* $\overline{p}$ *so that* $p\overline{p} \equiv 1\ (\mathcal{N})$. *Then*

$$\mathbb{E}_\sigma | T(p) = \sum_{r=0}^{n} \chi(p^{n-r}) p^{k(n-r)-(n-r)(n-r+1)/2} \beta_p(n,r) \mathbb{E}_{(p,\overline{p}^r)\cdot\sigma}.$$

**Idea of proof:**

The expression for $\mathbb{E}_\sigma(\tau)|T(p)$ is more complicated when $p \nmid \mathcal{N}$. Still, we know

$$\mathbb{E}_\sigma|T(p) = \sum_{\sigma'} c_{\sigma,\sigma'} \mathbb{E}_{\sigma'}$$

for some $c_{\sigma,\sigma'}$. So we show $c_{\sigma,\sigma'} \neq 0$ only if

$$(M_{(\overline{p},p^r)\cdot\sigma'} \;\; I) \in GL_n(\mathbb{Z})(M_\sigma \;\; I)\Gamma_0(\mathcal{N}).$$

For such $\sigma'$, we replace $\mathbb{E}_{\sigma'}$ by $\mathbb{E}_{(p,\overline{p}^r)\cdot\sigma}$ to make computations with the character easier, and we proceed as before. $\square$

**Corollary.** Take $\psi$ to be a character on $\mathcal{U}_\mathcal{N} \times \mathcal{U}_\mathcal{N}$.
(So $\psi(v, w) = \psi_1(v)\psi_2(w)$ for characters $\psi_1, \psi_2$ on $\mathcal{U}_\mathcal{N}$.)
Set

$$\mathbb{E}_{\sigma,\psi} = \sum_{v,w \in \mathcal{U}_\mathcal{N}} \overline{\psi}(v, w)\mathbb{E}_{(v,w)\cdot\sigma}.$$

Note that by orthogonality of characters, we have

$$\text{span}\{\mathbb{E}_\sigma\}_\sigma = \text{span}\{\mathbb{E}_{\sigma,\psi}\}_{\sigma,\psi}.$$

Then for prime $p \nmid \mathcal{N}$,

$$\mathbb{E}_{\sigma,\psi}|T(p) = \lambda_{\sigma,\psi}(p)\mathbb{E}_{\sigma,\psi}$$

where

$$\lambda_{\sigma,\psi}(p) = \psi_1(p)\overline{\psi}_2(p^n) \cdot \prod_{i=1}^{n}(\psi_2\chi(p)p^{k-i} + 1).$$

**Idea of proof:**

$$\mathbb{E}_{\sigma,\psi}|T(p) = \sum_{v,w,r} \overline{\psi}(v,w)\chi(p^{n-r})p^*\boldsymbol{\beta}_p(n,r)\mathbb{E}_{(pv,\overline{p}^r w)\cdot\sigma}.$$

Making the change of variables $v \mapsto \overline{p}v$, $w \mapsto p^r w$, we get

$$\mathbb{E}_{\sigma,\psi}|T(p) = \psi_1(p)\chi(p^n)p^{kn-n(n+1)/2}S(n,k)\,\mathbb{E}_{\sigma,\psi}$$

where

$$S(n,k) = \sum_{r=0}^{n} \psi_2\chi(\overline{p}^r)p^{-kr+r(r+1)/2}\boldsymbol{\beta}_p(n,r).$$

Using that $\boldsymbol{\beta}_p(n,r) = p^r\boldsymbol{\beta}_p(n-1,r) + \boldsymbol{\beta}_p(n-1,r-1)$, we find that

$$\begin{aligned} S(n,k) &= (\psi_2\chi(\overline{p})p^{1-k}+1)S(n-1,k-1) \\ &= \prod_{i=1}^{n} (\psi_2\chi(\overline{p})p^{i-k}+1). \end{aligned}$$

Then we collect and rearrange terms. □

Similar to the previous theorem, we have:

*Theorem*

*With $p$ a prime not dividing $\mathcal{N}$,*

$$\mathbb{E}_\sigma | T_j(p^2) = \boldsymbol{\beta}_p(n,j) \sum_{r+s\le j} \chi(p^{j-r+s}) p^{k(j-r+s)-(j-r)(n+1)} \cdot \boldsymbol{\beta}_p(j,r)\boldsymbol{\beta}_p(j-r,s)\,\mathrm{sym}_p(j-r-s)\mathbb{E}_{(1,p^{s-r})\cdot\sigma}$$

*(where $\mathrm{sym}_p(t)$ is the number of invertible symmetric $t \times t$ matrices modulo $p$).*
*Consequently $\mathbb{E}_{\sigma,\psi}$ is an eigenform for $T_j(p^2)$.*

Taking a different set of generators $T(p),\ T'_j(p^2)$ for the local Hecke algebra, we get that

$$\mathbb{E}_{\sigma,\psi} | T'_j(p^2) = \lambda'_{j;\sigma,\psi}(p^2)\, \mathbb{E}_{\sigma,\psi}$$

where

$$\lambda'_{j;\sigma,\psi}(p^2) = \boldsymbol{\beta}_p(n,j) p^{(k-n)j+j(j-1)/2} \chi(p^j) \prod_{i=1}^{j} (\psi_2 \chi(p) p^{k-i} + 1).$$

## Hecke operators on half-integral weight Siegel Eisenstein series: work in progress

We can build Siegel Eisenstein series of weight $m/2$, $m$ odd, by first defining

$$1(\tau)|\gamma = \left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^{-m}$$

where $\gamma \in \Gamma_0(4)$ and

$$\theta(\tau) = \sum_{U \in \mathbb{Z}^{1,n}} \exp(2\pi i Tr({}^tUU\tau)).$$

Then with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$,

$$\frac{\theta(\gamma\tau)}{\theta(\tau)} = \det(C\tau + D)^{1/2} \frac{\mathcal{G}_{-C}(D)}{(\det D)^{1/2}}$$

where

$$\mathcal{G}_{-C}(D) = \sum_{U \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}D} \exp(2\pi i Tr(-{}^tUUD^{-1}C)).$$

Using the theta series transformation, we get identities such as

$$\frac{\mathcal{G}_C(D)}{(\det D)^{1/2}} = \frac{\mathcal{G}_C(D+CY)}{(\det D+CY)^{1/2}}$$

for symmetric $Y \in \mathbb{Z}^{n,n}$. Then we can modify the preceding arguments to analyse Hecke operators on these Siegel theta series.

THANK YOU!