Lifts from two elliptic modular forms to Siegel modular forms of half-integral weight of even degrees

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Explicit Theory of Automorphic Forms
at Tongji University

Goal

Let k be an even integer. Then there exists a lift

$$\begin{array}{ccccc} S^+_{k-\frac{1}{2}} & \times & S^+_{k-n-\frac{1}{2}} & \to & S^{+(2n)}_{k-\frac{1}{2}} \\ \uplus & & \uplus & & \uplus \\ (f & , & g) & \mapsto & \mathcal{F}_{f,g} \end{array}$$

Here $S^+_{k-\frac{1}{2}}$ is the Kohnen plus space and $S^{+(2n)}_{k-\frac{1}{2}}$ is the generalized plus space of degree $2n$.

Notation 1

$$\mathfrak{H}_n := \left\{ \tau \in M_n(\mathbb{C}) \,|\, \tau = {}^t\tau,\ \mathrm{Im}(\tau) > 0 \right\},$$
$$\mathrm{Sp}(n,K) := \left\{ M \in \mathrm{GL}(2n,K) \,|\, M \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) {}^tM = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) \right\},$$

Sym_n^* : the set of all half-integral symmetric matrices with size n,

$$e(*) := exp(2\pi i\, tr(*)),$$

$$M_k^n := \{\text{Siegel modular form of weight } k \text{ of degree } n\},$$
$$S_k^n := \{\text{Siegel cusp form of weight } k \text{ of degree } n\}.$$

Notation 2

$$\Gamma_n \quad := \quad \mathrm{Sp}(n, \mathbb{Z})$$

$$\Gamma^J_{n,r} \quad := \quad \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1_r & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1_r \end{pmatrix} \in \Gamma_{n+r} \right\}.$$

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Let $\mathcal{M} \in \mathrm{Sym}_r^*$.

Definition (Jacobi forms of matrix index)

Let ϕ be a holomorphic function on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. Such ϕ is called a Jacobi form of weight k of index $\mathcal{M}$ of degree n, if

$$(\phi(\tau, z)\, e(\mathcal{M}\omega))\,|_k\gamma \quad = \quad \phi(\tau, z)\, e(\mathcal{M}\omega)$$

for any $\gamma \in \Gamma_{n,r}^J$ and for any $\left(\begin{smallmatrix} \tau & z \\ {}^t z & \omega \end{smallmatrix}\right) \in \mathfrak{H}_{n+r}$.
If $n = 1$, then ϕ is required to satisfy the cusp condition.

$$J_{k,\mathcal{M}}^{(n)} := \{\text{ Jacobi form of weight } k, \text{ index } \mathcal{M}, \text{ degree } n \}$$

Fourier-Jacobi expansion

Let $F \in M_k^{n+r}$. The expansion

$$F\left(\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix}\right) = \sum_{\mathcal{M} \in Sym_r^*} \phi_{\mathcal{M}}(\tau, z)\, e(\mathcal{M}\omega)$$

is called the Fourier-Jacobi expansion of F.

- We remark $\phi_{\mathcal{M}} \in J_{k,\mathcal{M}}^{(n)}$
- For Siegel modular forms of half-integral weight we can also consider the Fourier-Jacobi expansion.

Fourier-Jacobi expansion

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- We put

$$\theta(\tau) \;:=\; \sum_{p \in \mathbb{Z}^n} e(p\tau\, {}^t p) \qquad (\tau \in \mathfrak{H}_n).$$

- A holomorphic function F on $\mathfrak{H}_n$ is called a Siegel modular form of weight $k - \frac{1}{2}$ of degree n, if F satisfies

$$F(M \cdot \tau) \;=\; \left(\frac{\theta(M \cdot \tau)}{\theta(\tau)}\right)^{2k-1} F(\tau) \qquad \text{for any } M \in \Gamma_0^{(n)}(4).$$

  (+ cusp condition for $n = 1$.)

- $J^{(n)}_{k-\frac{1}{2},m} := \{$Jacobi forms of weight $k - \frac{1}{2}$ of index $m\}$.

The generalized plus space is defined by

$$M_{k-\frac{1}{2}}^{+(n)} := \left\{ f \in M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4)) \,|\, A_f(N) = 0 \text{ unless } N \equiv (-1)^{k+1} \lambda^t \lambda \mod 4 \text{ with some } \lambda \in \mathbb{Z}^n \right\},$$

where $A_f(N)$ is the N-th Fourier coefficient of f, i.e.

$$f(\tau) = \sum_{N \in Sym_n^*} A_f(N)\, e(N\tau).$$

- $M_{k-\frac{1}{2}}^{+(1)}$ is the Kohnen plus space.

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Generalized Plus space and the isomorphism

Theorem (Eichler-Zagier ($n = 1$), Ibukiyama ($n > 1$))

If k is an even integer, then

$$M^{+(n)}_{k-\frac{1}{2}} \cong J^{(n)}_{k,1}$$

as Hecke algebra module

Several generalizations of this theorem are known
(skew-holomorphic Jacobi forms: Skoruppa, Arakawa, H.
with character: Ibukiyama-H.
vector valued: Kimura, Ibukiyama
etc.)

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Plus space for Jacobi forms

Let $\phi \in J^{(n)}_{k-\frac{1}{2},m}$ be a Jacobi form of weight $k-\frac{1}{2}$ of index m $(\in \mathbb{Z})$.

Fourier expansion of $\phi(\tau,z)e(m\omega)$:

$$\phi(\tau,z)e(m\omega) = \sum_{\substack{N\in Sym_n^*,\, R\in\mathbb{Z}^n \\ 4Nm-R^tR\geq 0}} A\left(\begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & m \end{pmatrix}\right) e\left(\begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & m \end{pmatrix}\begin{pmatrix} \tau & z \\ {}^tz & \omega \end{pmatrix}\right)$$

Definition (Plus space for Jacobi forms)

$\phi \in J^{+(n)}_{k-\frac{1}{2},m}$ if and only if

$$A(M) = 0 \text{ unless } M \equiv (-1)^{k+1}\lambda^t\lambda \mod 4 \text{ with some } \lambda \in \mathbb{Z}^{n+1}$$

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Let $\mathcal{M} = \begin{pmatrix} l & \frac{1}{2}r \\ \frac{1}{2}r & 1 \end{pmatrix}$ be a half-integral symmetric matrix of size 2. We put

$$m = \det 2\mathcal{M}.$$

Proposition

If k is an even integer, then

$$J^{+(n)}_{k-\frac{1}{2},m} \cong J^{(n)}_{k,\mathcal{M}}$$

as Hecke algebra modules.

$$J^{+(n)}_{k-\frac{1}{2},m} \subset J^{(n)}_{k-\frac{1}{2},m} \ : \text{ plus space for Jacobi forms}$$

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Main result

Theorem

Let k be an even integer. Let $f \in S^+_{k-\frac{1}{2}}$ and $g \in S^+_{k-n-\frac{1}{2}}$ be Hecke eigenforms in the Kohnen plus-spaces. Then there exists $\mathcal{F}_{f,g} \in S^{+(2n)}_{k-\frac{1}{2}}$. Under the assumption $\mathcal{F}_{f,g} \not\equiv 0$, the form $\mathcal{F}_{f,g}$ is a Hecke eigenform with the (modified) Zhuravlev L-function

$$L(s, \mathcal{F}_{f,g}) \;=\; L(s,f) \prod_{i=1}^{2n-1} L(s-i, g).$$

Here the L-function is defined by

$$L(s, \mathcal{F}) \;:=\; \prod_{p} \prod_{i=1}^{n} \left\{ (1 - \alpha_{i,p} p^{k-\frac{3}{2}-s})(1 - \alpha_{i,p}^{-1} p^{k-\frac{3}{2}-s}) \right\}^{-1}$$

for Hecke eigenform $\mathcal{F} \in M^{+(n)}_{k-\frac{1}{2}}$. ($\{\alpha^{\pm 1}_{i,p}\}_i$ is the p-parameter of $\mathcal{F}$.)

The construction of the lift $\mathcal{F}_{f,g}$

$(f, g) \in S^+_{k-\frac{1}{2}} \otimes S^+_{k-n-\frac{1}{2}}$.

$g \in S^+_{k-n-\frac{1}{2}} \xrightarrow{\text{Ikeda lift}} I(g) \in S_k^{2n+2}$

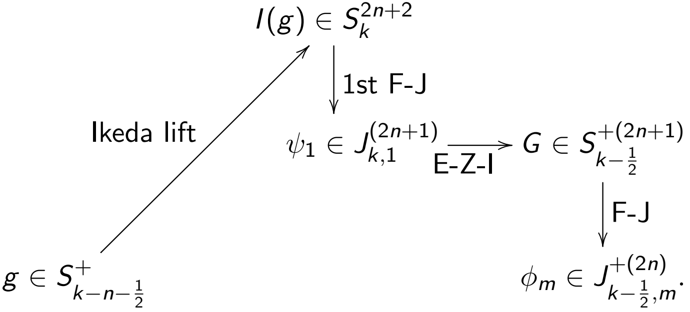
$I(g) \in S_k^{2n+2} \xrightarrow{\text{1st F-J}} \psi_1 \in J_{k,1}^{(2n+1)}$

$\psi_1 \in J_{k,1}^{(2n+1)} \xrightarrow{\text{E-Z-I}} G \in S^{+(2n+1)}_{k-\frac{1}{2}}$

For $\tau \in \mathfrak{H}_{2n}$ we set

$$\mathcal{F}_{f,g}(\tau) := \frac{1}{6} \int_{\Gamma_0(4)\backslash\mathfrak{H}_1} G\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)\right) \overline{f(\omega)}\, Im(\omega)^{k-\frac{5}{2}}\, d\omega \in S^{+(2n)}_{k-\frac{1}{2}}.$$

$(f,g) \in S^+_{k-\frac{1}{2}} \otimes S^+_{k-n-\frac{1}{2}}$.



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$(f,g) \in S^+_{k-\frac{1}{2}} \otimes S^+_{k-n-\frac{1}{2}}$.

$$\begin{array}{ccccc} & & I(g) \in S_k^{2n+2} & & \\ & \nearrow^{\text{Ikeda lift}} & \big\downarrow \text{1st F-J} & & \\ & & \psi_1 \in J_{k,1}^{(2n+1)} & \xrightarrow[\text{E-Z-I}]{} & G \in S^{+(2n+1)}_{k-\frac{1}{2}} \\ & & \big\downarrow \text{F-J} & & \big\downarrow \text{F-J} \\ g \in S^+_{k-n-\frac{1}{2}} & & \psi_{\mathcal{M}} \in J^{(2n)}_{k,\mathcal{M}} & \longrightarrow & \phi_m \in J^{+(2n)}_{k-\frac{1}{2},m}. \end{array}$$

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$$\mathcal{F}_{f,g}(\tau) \quad := \quad \frac{1}{6} \int_{\Gamma_0(4)\backslash\mathfrak{H}_1} G\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)\right) \overline{f(\omega)}\, Im(\omega)^{k-\frac{5}{2}}\, d\omega \in S^{+(2n)}_{k-\frac{1}{2}}.$$

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Maass relation

Let

$$F(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \omega \end{smallmatrix}\right)) = \sum_{m\in\mathbb{Z}} \phi_m(\tau, z)\, e(m\omega) \in M_k^2$$

be a Fourier-Jacobi expansion.
The Maass relation is a relation among Fourier-Jacobi coefficients

$$\boxed{\phi_m = \phi_1 | V_m}$$

for any non-negative integer m. Here V_m is an index-shift map

$$V_m \ : \ J_{k,t}^{(1)} \to J_{k,tm}^{(1)}.$$

- We have also $\phi_m | V_p = \phi_{mp} + p^{k-1}\phi_{\frac{m}{p}}$ for prime p.

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- We have also $\phi_m | V_p = \phi_{mp} + p^{k-1}\phi_{\frac{m}{p}}$ for prime p.

The map V_m is defined by

$$(\phi|V_m)(\tau,z) := m^{k-1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_1\backslash M_2(\mathbb{Z}) \\ ad-bc=m}} (c\tau+d)^{-k} e\left(ml\frac{cz^2}{c\tau+d}\right) \times \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{mz}{c\tau+d}\right)$$

for $\phi \in J_{k,t}^{(1)}$, $(\tau,z) \in \mathfrak{H}_1 \times \mathbb{C}$. Then $\phi|V_m \in J_{k,tm}^{(1)}$.

- $\phi(\tau,0) \in M_k^1$
- $(\phi|V_m)(\tau,0) = \phi(\tau,0)|T(m)$ with the usual Hecke operator $T(m)$ which acts on the space of elliptic modular forms.

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&\times \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{mz}{c\tau+d} \right)
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- $(\phi|V_m)(\tau,0) = \phi(\tau,0)|T(m)$ with the usual Hecke operator $T(m)$ which acts on the space of elliptic modular forms.

Generalization of V_m

For any prime p and for i $(0 \leq i \leq n)$ we can define a map

$$\tilde{V}_{i,n-i}(p^2) \, : \, J^{+(n)}_{k-\frac{1}{2},m} \to J^{+(n)}_{k-\frac{1}{2},mp^2}$$

- $\phi(\tau, 0) \in M^{+(n)}_{k-\frac{1}{2}}$ for any $\phi \in J^{+(n)}_{k-\frac{1}{2},m}$.
- $(\phi|\tilde{V}_{i,n-i}(p^2))(\tau, 0) = \phi(\tau, 0)|\tilde{T}_{i,n-i}(p^2)$ with the Hecke operator $\tilde{T}_{i,n-i}(p^2) : M^{+(n)}_{k-\frac{1}{2}} \to M^{+(n)}_{k-\frac{1}{2}}$.

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Index-shift maps and the isomorphism

Proposition

We have the commutative diagram

$$\begin{array}{ccc} J^{(n)}_{k,\mathcal{M}} & \longrightarrow & J^{(n)+}_{k-\frac{1}{2},m} \\ {\scriptstyle V,U}\downarrow & & \downarrow{\scriptstyle \tilde{V},\tilde{U}} \\ J^{(n)}_{k,\mathcal{M}\left[\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right]} & \longrightarrow & J^{(n)+}_{k-\frac{1}{2},mp^2}. \end{array}$$

where V, U, $\tilde{V}$ and $\tilde{U}$ are index-shift maps.

Generalized Maass relation

Let

$$G\left(\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix}\right) = \sum_{m\in\mathbb{Z}} \phi_m(\tau, z) e(m\omega)$$

be the Fourier-Jacobi expansion of G $(\in S^{+(2n+1)}_{k-\frac{1}{2}})$

Theorem (Generalized Maass relation)

$$\begin{aligned} & \phi_m | (\tilde{V}_{0,2n}(p^2), \tilde{V}_{1,2n-1}(p^2), \ldots, \tilde{V}_{2n,0}(p^2)) \\ = \ & \left(\phi_{\frac{m}{p^2}} | U_{p^2}, \ \phi_m | U_p, \ \phi_{mp^2}\right) A(\beta_p) \end{aligned}$$

for any prime p. (The both sides are vectors of forms.)

$A(\beta_p)$ is a certain $3 \times (2n+1)$ matrix which depends on the choice of $g \in S^{+}_{k-n-\frac{1}{2}}$. $(\{\beta_p^{\pm 1}\}_p$ is the Satake parameter of g.)

Sketch of proof of main theorem

$$(\phi_m|\tilde{V}_{i,2n-i}(p^2))(\tau,0) = \phi_m(\tau,0)|\tilde{T}_{i,2n-i}(p^2)$$

$$\begin{aligned}
& G\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|\tilde{T}_{i,2n-i}(p^2) \\
= & \sum_m \left(\phi_m(\tau,0)|\tilde{T}_{i,2n-i}(p^2)\right) e(m\omega) \\
= & \sum_m \left(\phi_m|\tilde{V}_{i,2n-i}(p^2)\right)(\tau,0) e(m\omega) \\
= & \sum_m \left(\text{lin. comb. of } \phi_{\frac{m}{p^2}}(\tau,0),\ \phi_m(\tau,0),\ \phi_{mp^2}(\tau,0)\right) e(m\omega) \\
= & \text{ lin. comb. of } G\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right) \text{ and } G\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\omega T^{+}(p^2).
\end{aligned}$$

$$\begin{aligned}
& \mathcal{F}_{f,g}|\tilde{T}_{i,2n-i}(p^2) \\
& = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathfrak{H}_1} \left(G\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|\tilde{T}_{i,2n-i}(p^2)\right)\overline{f(\omega)}\, Im(\omega)^{k-\frac{5}{2}}\, d\omega = c\,\mathcal{F}_{f,g}.
\end{aligned}$$

Sketch of proof of generalized Maass relation

- Consider $J_{k,\mathcal{M}}^{(n)}$ instead of $J_{k-\frac{1}{2},m}^{+(n)}$ (Here $\mathcal{M} = \begin{pmatrix} l & \frac{1}{2}r \\ \frac{1}{2}r & 1 \end{pmatrix}$).
- Introduce V-map for $J_{k,\mathcal{M}}^{(n)}$. $V_{i,n-i}(p^2) : J_{k,\mathcal{M}}^{(n)} \to J_{k,\mathcal{M}\left[\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right]}^{(n)}$
- Show a generalized Maass relation for Siegel-Eisenstein series. (Fourier-Jacobi expansion of S-E series with respect to $\mathcal{M}$.) Ikeda lifts inherit some properties of Siegel-Eisenstein series.
- Use the relation among Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series (w.r.t. index $\mathcal{M}$) (It follows from Boecherer's result.)
- Use "Zharvkovskaya's theorem" for Jacobi forms of index $\mathcal{M}$. (This is a commutative relation between Siegel ϕ-operator on Jacobi forms and V-maps. (Generalization of Zharkovskaya and Krieg's results.))

$$
\begin{array}{ccccc}
 & & E_k^{(2n+2)} \in M_k^{2n+2} & & \\
 & \nearrow & \downarrow \text{1st F-J} & & \\
 & & e_{k,1}^{(2n+1)} \in J_{k,1}^{(2n+1)} & \xrightarrow[\text{E-Z-I}]{} & H_{k-\frac{1}{2}}^{(2n+1)} \in M_{k-\frac{1}{2}}^{+(2n+} \\
 & \text{Ikeda lift} & \downarrow \text{F-J} & & \downarrow \text{F-J} \\
 & & e_{k,\mathcal{M}}^{(2n)} \in J_{k,\mathcal{M}}^{(2n)} & \longrightarrow & e_{k-\frac{1}{2},m}^{(2n)} \in J_{k-\frac{1}{2},m}^{+(2n)} \\
 & \nearrow & & & \\
H_{k-n-\frac{1}{2}}^{(1)} \in M_{k-n-\frac{1}{2}}^{+} & & & &
\end{array}
$$

Proposition

Let $\mathcal{M} = \begin{pmatrix} l & \frac{1}{2}r \\ \frac{1}{2}r & 1 \end{pmatrix}$ *and* $\Phi : J^{(n)}_{k,\mathcal{M}} \to J^{(n-1)}_{k,\mathcal{M}}$.

For any Jacobi form $\phi \in J^{(n)}_{k,\mathcal{M}}$ *and for any prime p, we have*

$$\Phi(\phi|V_{i,n-i}(p^2)) = \Phi(\phi)|V_{i,n-i}(p^2)^*,$$

where $V_{i,n-i}(p^2)^*$ *is a map* $V_{i,n-i}(p^2)^* : J^{(n-1)}_{k,\mathcal{M}} \to J^{(n-1)}_{k,\mathcal{M}\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]}$ *given by*

$$\begin{aligned} V_{i,n-i}(p^2)^* &= p^{i+2-k} V_{i,n-i-1}(p^2) \\ &\quad + p(1 + p^{2n+1-2k}) V_{i-1,n-i}(p^2) \\ &\quad + (p^{2n-2i+2} - 1) p^{i-k} V_{i-2,n-i+1}(p^2). \end{aligned}$$

Thank you!