On the representation numbers of ternary quadratic forms and modular forms of weight 3/2

Hourong Qin
(joint with Xuejun Guo and Yuzhen Peng)

Department of Mathematics, Nanjing University, China

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Theorem (Fermat)

An odd prime p is expressible as $p = x^2 + y^2$ with x and y integers, if and only if $p \equiv 1 \mod 4$.

By this theorem, one easily proves that a number N can be represented as a sum of two squares precisely when N is of the form

$$n^2 \prod p_i$$

where p_i is 2 or a prime congruent to $1 \mod 4$.
Albert Girard first made the observation in 1632. Fermat announced this theorem in a letter to Marin Mersenne dated December 25, 1640; for this reason this theorem is sometimes called Fermat's Christmas Theorem.

Representation number of two squares

Let

$$n = 2^{a_0} m^2 p_1^{a_1} \cdots p_k^{a_k} > 1,$$

where the prime divisor of m is congruent to 3 modulo 4 and p_i is is congruent to 1 modulo 4. Then the number of ways to represent n as the sum of two squares is given by

$$r_2(n) = 4(a_1 + 1) \cdots (a_k + 1).$$

In the following context, we will use $r_t(n)$ for the number of ways to represent n as the sum of t squares.

Legendre's three-square theorem

Theorem

Any natural number that is not of the form

$$n = 4^a(8b + 7)$$

for integers a and b can be represented as the sum of three integer squares:

$$n = x^2 + y^2 + z^2.$$

This theorem was stated by Adrien-Marie Legendre in 1798. His proof was incomplete, leaving a gap which was later filled by Carl Friedrich Gauss.

Representation number of three squares

Gauss proved that if n is square free and $n > 4$, then

$$r_3(n) = \begin{cases} 24h(-n), & \text{if } n \equiv 3 \bmod 8; \\ 12h(-4n), & \text{if } n \equiv 1,\ 2,\ 5,\ 6 \bmod 8; \\ 0, & \text{if } n \equiv 7 \bmod 8, \end{cases}$$

where $h(d)$ is the class number of the unique order of $\mathbb{Q}(\sqrt{d})$ with discriminant d.

Lagrange's four-square theorem

Theorem

Any natural number can be represented as the sum of four integer squares

$$n = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

where the four numbers $a_0, \ a_1, \ a_2, \ a_3$ *are integers.*

The theorem appears in the Arithmetica of Diophantus, translated into Latin by Bachet in 1621 but was not proven until 1770 by Lagrange.

Representation number of four squares

The number of ways to represent n as the sum of four squares is eight times the sum of the divisors of n if n is odd and 24 times the sum of the odd divisors of n if n is even, i.e.

$$r_4(n) = \begin{cases} 8 \sum_{m \mid n} m, & \text{if } n \text{ is odd;} \\ 24 \sum_{\substack{m \mid n \\ m \text{ odd}}} m, & \text{if } n \text{ is even.} \end{cases}$$

Equivalently, it is eight times the sum of all its divisors which are not divisible by 4, i.e.

$$r_4(n) = 8 \sum_{m \,:\, 4 \nmid m \mid n} m.$$

In particular, for a prime number p we have the explicit formula

$$r_4(p) = 8(p+1).$$

Representation number of five squares

The number of ways to represent n as the sum of five squares is

$$r_5(n) = \begin{cases} -80\sum_{j=1}^{(n-1)/2} \left(\frac{j}{n}\right) j, & \text{if } n \equiv 1 \bmod 8,\ \ n \neq 1; \\ -80\sum_{j=1}^{(n-1)/2} (-1)^j \left(\frac{j}{n}\right) j, & \text{if } n \equiv 3 \bmod 8; \\ -112\sum_{j=1}^{(n-1)/2} (-1)^j \left(\frac{j}{n}\right) j, & \text{if } n \equiv 5 \bmod 8; \\ 80\sum_{j=1}^{(n-1)/2} (-1)^j \left(\frac{j}{n}\right) j, & \text{if } n \equiv 7 \bmod 8; \end{cases}$$

$$r_5(2n) = 80n \sum_{j=1,\ j\equiv 1 \bmod 4}^{n-1} \left(\frac{2}{j}\right)\left(\frac{j}{m}\right) + 80(-1)^{(n-1)/2}$$

$$\sum_{j=1,\ j\equiv 3 \bmod 4}^{n-1} \left(\frac{2}{j}\right)\left(\frac{j}{m}\right)(m-j) \text{ for odd } n > 1.$$

Representation number of six squares

The number of ways to represent n as the sum of six squares is

$$r_6(n) = 16\sum_{d|n}\chi(n/d)d^2 - 4\sum_{d|n}\chi(d)d^2$$

where

$$\chi(d) = \begin{cases} 1, & \text{if } d \equiv 1 \bmod 4; \\ -1, & \text{if } d \equiv -1 \bmod 4 \\ 0, & \text{if } d \equiv 0 \bmod 2. \end{cases}$$

Hurwitz Theorem

Theorem (Hurwitz)

Let n be a positive integer with prime factorization

$$n = 2^{e_2} \prod_{p \text{ odd}} p^{e_p}. \tag{1}$$

Then

$$r_3(n^2) = 6 \prod_{p \text{ odd}} \left(\frac{p^{e_p+1} - 1}{p-1} - \left(\frac{-1}{p} \right) \frac{p^{e_p} - 1}{p-1} \right),$$

where $\left(\frac{a}{p} \right)$ *is the Legendre symbol.*

Cooper and Lam

Recently Cooper and Lam established analogues of Hurwitz's formula for the cases

$$(a, b, c) = (1, 1, 2), (1, 1, 3), (1, 2, 2) \text{ and } (1, 3, 3).$$

For example:

Cooper and Lam Theorem

Theorem (Cooper and Lam)

Let n be a positive integer with prime factorization

$$n = 2^{e_2} \prod_{p \text{ odd}} p^{e_p}. \tag{2}$$

Then the number of $(x,\ y,\ z) \in \mathbb{Z}^3$ such that

$$n^2 = x^2 + 3y^2 + 3z^2$$

is given by

$$2(2^{e_2+2} - 3) \prod_{p\ \geq 5} \left(\frac{p^{e_p+1} - 1}{p - 1} - \left(\frac{-1}{p}\right) \frac{p^{e_p} - 1}{p - 1} \right),$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol.

Notations

Let $Q(x, y, z)$ be a positive definite ternary quadratic form with integer coefficients. We denote by $R_Q(n)$ the representation number of the integer n by Q, that is, the number of integral solutions (x, y, z) of the equation $Q(x, y, z) = n$. In particular, if

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + ryz + szx + txy,$$

we write $R_{(a,b,c,r,s,t)}(n)$ for $R_Q(n)$. If $r = s = t = 0$, we will also write $R_{(a,b,c,r,s,t)}(n)$ simply as $R_{(a,b,c)}(n)$. In particular, we write

$$S(n) = R_{(1,1,1,0,0,0)}(n)$$

for short.

Conjecture (Cooper and Lam 1)

Let n be a positive integer with prime factorization given by (1.4). Then when (b, c) takes the values in Table 1, the representation number $S_{(1,b,c)}(n^2)$ is a product of two integers:

$$S_{(1,b,c)}(n^2) = G(b, c, n) \cdot H(b, c, n),$$

where

$$H(b, c, n) = \prod_{p \nmid 2bc} \left(\frac{p^{e_p+1} - 1}{p - 1} - \left(\frac{-bc}{p} \right) \frac{p^{e_p} - 1}{p - 1} \right),$$

and

$$G(b, c, n) = \prod_{p \mid 2bc} g(b, c, p, e_p),$$

in which $g(b, c, p, e_p)$ has to be determined on an individual and case-by-case basis.

Table 1

Data for Cooper and Lam 1

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 13, 16, 22, 40, 70
3	3, 4, 5, 6, 9, 10, 12, 18, 21, 30, 45
4	4, 6, 8, 12, 24
5	5, 8, 10, 13, 25, 40
6	6, 9, 16, 18, 24
8	8, 10, 13, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

Hecke operator

Recall that Ramanujan's theta function, denoted by $\varphi(q)$, is defined to be

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \text{ where } q = e^{2\pi i z}.$$

Let b and c be fixed integers, the *Hecke operator* $T_{p^2}(b,c)$ is defined for any prime $p \nmid 2bc$ by

$$T_{p^2}(b,c)(\sum_{j=0}^{\infty} c_j q^j) = \sum_{j=0}^{\infty} c_{p^2 j} q^j + \sum_{j=0}^{\infty} (\left(\frac{-bcj}{p}\right)) c_j q^j + \sum_{j=0}^{\infty} c_j q^{p^2 j},$$

where $\left(\frac{-bcj}{p}\right)$ is the Legendre symbol.

Conjecture (Cooper and Lam 2)

Let b and c take any of the values given in Table 2. Then for any prime p with $p \nmid 2bc$, we have

$$T_{p^2}(b,c)(\varphi(q)\varphi(q^b)\varphi(q^c)) = (p+1)\varphi(q)\varphi(q^b)\varphi(q^c).$$

Table 2

Data for Conjecture Cooper and Lam 2

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 16
3	3, 4, 6, 9, 10, 12, 18, 30
4	4, 6, 8, 12, 24
5	5, 8, 10, 25, 40
6	6, 9, 16, 18, 24
8	8, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

Theorem (Berkovich and Jagy)

For any natural number n,

$$S(9n) - 3S(n) = 2R_{(1,1,3,0,0,1)}(n) - 4R_{(4,3,4,0,4,0)}(n),$$
$$S(25n) - 5S(n) = 4R_{(2,2,2,-1,1,1)}(n) - 8R_{(7,8,8,-4,8,8)}(n).$$

Berkovich and Jagy continued to investigate the value of $S(p^2n) - pS(n)$ for arbitrary odd prime p. They constructed two genera $TG_{1,p}$ and $TG_{2,p}$, where $TG_{1,p}$ consists of all the ternary quadratic forms with discriminant p^2, while $TG_{2,p}$ is the set of ternary quadratic forms $ax^2 + by^2 + cz^2 + ryz + szx + txy$ with discriminant $16p^2$ satisfying two conditions, namely, r, s, t are even and

$$R_{(a,b,c,r,s,t)}(n) = 0, \quad n \equiv 1, 2 \pmod 4. \tag{3}$$

Then the generalization of the above Theorem reads as follows :

Theorem (Berkovich and Jagy)

Let p be an odd prime. Then for any natural number n,

$$S(p^2 n) - pS(n) = 48 \sum_{Q \in TG_{1,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|} - 96 \sum_{Q \in TG_{2,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|},$$

where $\mathrm{Aut}(Q)$ is the finite group of integral automorphs of Q, and a sum over forms in a genus is understood to be the finite sum resulting from taking a single representative from each equivalence class of forms.

Main results 1

Table 3

Cooper and Lam 1 holds for the following cases (in black)

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 13, 16, 22, 40, 70
3	3, 4, 5, 6, 9, 10, 12, 18, 21, 30, 45
4	4, 6, 8, 12, 24
5	5, 8, 10, 13, 25, 40
6	6, 9, 16, 18, 24
8	8, 10, 13, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

Main results 2

Cooper and Lam 2 holds for all cases of Table 2.

Main results 3

We prove that Sun's conjecture holds, i.e.,

$$S_{(1,1,3)}(p) = \begin{cases} 12h(-3p), & \text{if } p \equiv 1 \pmod 8, \\ 8h(-3p), & \text{if } p \equiv 5 \pmod 8, \\ 2h(-3p), & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

and

$$S_{(1,1,3)}(3p) = \begin{cases} 4h(-p), & \text{if } p \equiv 1 \pmod 4, \\ 24h(-p), & \text{if } p \equiv 3 \pmod 8, \\ 16h(-p), & \text{if } p \equiv 7 \pmod 8, \end{cases}$$

where $h(d)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$.

Main results 4

We give a "global" proof of Berkovich and Jagy's proof.

Recall that the matrix associated to
$Q = ax^2 + by^2 + cz^2 + ryz + szx + txy$ is

$$A = \begin{pmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{pmatrix}.$$

The discriminant of Q is defined to be $\det(A)/2$, the level of Q is the minimal positive integer N such that NA^{-1} is an integral matrix with even diagonal entries, and the class number of Q is the number of equivalent classes in the genus of Q.
It is well known that

$$\theta_Q(z) = \sum_{n \geq 0} R_Q(n) e^{2\pi inz}$$

is a holomorphic function in the complex upper half-plane.

Furthermore $\theta_Q(z)$ is a modular form of weight $3/2$ with level N and character $\chi = (\frac{2\det(A)}{\cdot})$.
Denote by $\mathscr{M}(N,\omega)$ the complex linear space of modular forms of weight $3/2$, level N and character ω. Let $\mathscr{S}(N,\omega)$ be the subspace of cusp forms in $\mathscr{M}(N,\omega)$ and $\mathscr{E}(N,\omega)$ the orthogonal complement of $\mathscr{S}(N,\omega)$ in $\mathscr{M}(N,\omega)$ with respect to Petersson inner product.

It is clear that, if we can get an explicit expression of the function $\theta_Q(z)$, then we immediately derive an explicit formula for the representation number $R_Q(n)$. We note that, Pei constructed for some particular N and ω, an explicit basis for the space $\mathscr{E}(N, \omega)$. By virtue of these results we are able to reprove Berkovich and Jagy's identity and solve several conjectures proposed in Cooper & Lam and Sun.

Example: $Q(x,\ y,\ z) = x^2 + 5y^2 + 5z^2$

In this case, $\theta_Q(z) \in \mathscr{E}(20, \mathrm{id})$ and $\mathscr{S}(20, \mathrm{id}) = 0$, where id is the trivial character. Hence if one can get the values of $\theta_Q(z)$ at all cusps, one can get the explicit coordinate of $\theta_Q(z) \in \mathscr{E}(20, \mathrm{id})$. Pei proved that

$$S_{(1,5,5)}(m) = 2\pi\sqrt{m} \cdot \lambda(m, 20) \cdot \alpha(m) \cdot \left(A(5, m) + \frac{1}{5} \right).$$

The meaning of three functions $\lambda,\ \alpha$ and A will be explained in next two sides.

For any two integers a and b, let $\left(\frac{a}{b}\right)$ be the *Kronecker symbol*. For any fixed nonzero integer t, we can define a function χ_t on the integers as follows: Let $t = qs^2$ with q the square-free part of t. Then define $\chi_t = \left(\frac{q}{\bullet}\right)$ when $q \equiv 1 \pmod 4$ and $\chi_t = \left(\frac{4q}{\bullet}\right)$ when $q \equiv 2, 3 \pmod 4$. One sees that χ_t is a quadratic Dirichlet character. Denote by $\sigma(m, 4D)$ the sum $\sum \frac{\mu(a)\chi_{-m}(a)}{ab}$, where μ is the Möbius function, and the summation ranges over all the positive integers a, b that are both coprime to $4D$ and satisfy $(ab)^2 \mid m$.

Lemma

Let u be the square-free part of ℓ. Then

$$\sigma(\ell n^2, 4D) = \frac{1}{n}\left(\prod_{p \mid 4D} p^{e_p}\right) \cdot \left(\prod_{p \nmid 4D} \left(\frac{p^{e_p+1} - 1}{p-1} - \left(\frac{-u}{p}\right) \cdot \frac{p^{e_p} - 1}{p-1}\right)\right).$$

Recall that $L_k(s,\ \chi) = \sum_{(n,k)=1} \chi(n) n^{-s}$.

The function $\lambda(m, 4D)$. Put

$$\lambda(m, 4D) := \frac{L_{4D}(1, \chi_{-m})}{L_{4D}(2, \mathrm{id})} \cdot \sigma(m, 4D),$$

where id denotes the trivial character mod $4D$.

The functions $\alpha(m)$. For any prime p, let $h_p(m)$ be the natural number such that $p^{h_p(m)} \parallel m$ and $h'_p(m) := \frac{m}{p^{h_p(m)}}$ the p'-part of m. Define

$$\alpha(m) = \begin{cases} 3 \cdot 2^{-\frac{1+h_2(m)}{2}}, & \text{if } h_2(m) \text{ is odd}, \\ 3 \cdot 2^{-1-\frac{h_2(m)}{2}}, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 1 \pmod 4, \\ 2^{-\frac{h_2(m)}{2}}, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 3 \pmod 8, \\ 0, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 7 \pmod 8. \end{cases}$$

The functions $A(p, m)$.

$$A(p,m) = \begin{cases} p^{-1} - (1+p)p^{-\frac{3+h_p(m)}{2}}, & \text{if } h_p(m) \text{ is odd,} \\ p^{-1} - 2p^{-1-\frac{h_p(m)}{2}}, & \text{if } h_p(m) \text{ is even and } \left(\frac{-h_p'(m)}{p}\right) \\ p^{-1}, & \text{if } h_p(m) \text{ is even and } \left(\frac{-h_p'(m)}{p}\right) \end{cases}$$

where p is an odd prime.

Example: $Q(x,\ y,\ z) = x^2 + 5y^2 + 5z^2$

Let n be a positive integer with the prime factorization

$$n = 2^{e_2} \prod_{p \text{ odd}} p^{e_p}. \tag{4}$$

We denote by $H_{(a,b,c)}(n)$ the following product

$$H_{(a,b,c)}(n) = \prod_{p \nmid 2abc} \left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-abc}{p}\right) \frac{p^{e_p}-1}{p-1} \right).$$

Then

$$S_{(1,5,5)}(n^2) = 2 \cdot 5^{e_5} H_{(1,5,5)}(n).$$

Table 4

(a,b,c)	$S_{(a,b,c)}(n^2)$
$(1,1,1)$	$6H_{(1,1,1)}(n)$
$(1,1,2)$	$\begin{cases} 4H_{(1,1,2)}(n), & \text{if } n \text{ is odd,} \\ 12H_{(1,1,2)}(n), & \text{if } n \text{ is even} \end{cases}$
$(1,1,3)$	$4(2^{e_2+1}-1)H_{(1,1,3)}(n)$
$(1,1,4)$	$\begin{cases} 4H_{(1,1,4)}(n), & \text{if } n \text{ is odd,} \\ 6H_{(1,1,4)}(n), & \text{if } n \text{ is even} \end{cases}$
$(1,1,5)$	$2(5^{e_5+1}-3)H_{(1,1,5)}(n)$
$(1,1,6)$	$\begin{cases} 4H_{(1,1,6)}(n), & \text{if } n \text{ is odd,} \\ 4(2^{e_2+1}-3)H_{(1,1,6)}(n), & \text{if } n \text{ is even} \end{cases}$
$(1,1,8)$	$\begin{cases} 4H_{(1,1,8)}(n), & \text{if } n \text{ is odd,} \\ 4H_{(1,1,8)}(n), & \text{if } n \equiv 2 \pmod 4, \\ 12H_{(1,1,8)}(n), & \text{if } n \equiv 0 \pmod 4 \end{cases}$
$(1,2,2)$	$\begin{cases} 2H_{(1,2,2)}(n), & \text{if } n \text{ is odd,} \\ 6H_{(1,2,2)}(n), & \text{if } n \text{ is even} \end{cases}$
$(1,2,3)$	$\begin{cases} 2(3^{e_3+1}-2)H_{(1,2,3)}(n), & \text{if } n \text{ is odd,} \\ 6(3^{e_3+1}-2)H_{(1,2,3)}(n), & \text{if } n \text{ is even.} \end{cases}$
$(1,2,4)$	$\begin{cases} 2H_{(1,2,4)}(n), & \text{if } n \text{ is odd,} \\ 4H_{(1,2,4)}(n), & \text{if } n \equiv 2 \pmod 4, \\ 12H_{(1,2,4)}(n), & \text{if } n \equiv 0 \pmod 4 \end{cases}$

Table 4-continued

(a, b, c)	$S_{(a,b,c)}(n^2)$
$(1, 2, 6)$	$2(3^{e_3+1} - 2)H_{(1,2,6)}(n)$
$(1, 3, 3)$	$2(2^{e_2+2} - 3)H_{(1,3,3)}(n)$
$(1, 3, 6)$	$2 \cdot 3^{e_3} H_{(1,3,6)}(n)$, if n is odd, $2 \cdot 3^{e_3+1} H_{(1,3,6)}(n)$, if n is even
$(1, 4, 4)$	$2H_{(1,4,4)}(n)$, if n is odd, $6H_{(1,4,4)}(n)$, if n is even
$(1, 4, 8)$	$2H_{(1,4,8)}(n)$, if n is odd, $4H_{(1,4,8)}(n)$, if $n \equiv 2 \pmod 4$, $12H_{(1,4,8)}(n)$, if $n \equiv 0 \pmod 4$
$(1, 5, 5)$	$2 \cdot 5^{e_5} H_{(1,5,5)}(n)$
$(1, 6, 6)$	$2H_{(1,6,6)}(n)$, if n is odd, $2(2^{e_2+1} - 3)H_{(1,6,6)}(n)$, if n is even
$(2, 2, 3)$	$4(2^{e_2} - 1)H_{(2,2,3)}(n)$
$(2, 3, 3)$	0
$(2, 3, 6)$	$2(3^{e_3} - 1)H_{(2,3,6)}(n)$, if n is odd, $6(3^{e_3} - 1)H_{(2,3,6)}(n)$, if n is even

Let $\mathfrak{G}$ be the genus containing the ternary quadratic form Q. Recall that the *mass* of $\mathfrak{G}$ is by definition

$$M(\mathfrak{G}) := \sum_{Q \in \mathfrak{G}} \frac{1}{|\mathrm{Aut}(Q)|},$$

where the sum is over a complete representative system of equivalence classes of forms in $\mathfrak{G}$. Then the *theta series associated to* $\mathfrak{G}$ is defined to be

$$\theta_{\mathfrak{G}}(z) := \frac{1}{M(\mathfrak{G})} \sum_{Q \in \mathfrak{G}} \frac{\theta_Q(z)}{|\mathrm{Aut}(Q)|},$$

where $\theta_Q(z)$ is the usual theta series associated to the form Q. If Q is ternary form with discriminant d and level N then the function $\theta_{\mathfrak{G}}(z)$ is in the space $\mathscr{E}(N, \chi_d)$, and $\theta_{\mathfrak{G}}(z)$ has the same values as $\theta_Q(z)$ does at all cusps.

If the class number of $Q(x, y, z)$ is 1, then for any prime $p \nmid N$,

$$T_{p^2}(\theta_Q(z)) = (p+1)\theta_Q(z)$$

by the Lemma of Proposition 1 of [16].

[16]P.Ponomarev, Ternary quadratic forms and Shimura's correspondence, Nagoya Math. J. 81 (1981), 123–151.

All the ternary diagonal quadratic forms of Table 2 are of class number 1 . So Conjecture Cooper and Lam 2 is true. Moreover, Conjecture Cooper and Lam 2 is complete, that is, Table 2 contains exactly all the cases (b, c) such that

$$T_{p^2}(\theta_{x^2+by^2+cz^2}(z)) = (p+1)\theta_{x^2+by^2+cz^2}(z)$$

for any prime $p \nmid 2bc$.

Berkovich and Jagy's genus identity

Let p be an odd prime. Let $\theta(z) = \sum_{n=0}^{\infty} S(n)q^n$, where $q = e^{2\pi iz}$. Then $\theta(z)$ is in the space $\mathscr{E}(4, \mathrm{id}) \subset \mathscr{E}(4p, \mathrm{id})$, where we use "id" for the trivial character. Then function $\psi_p(z) = \sum_{n=0}^{\infty} S(p^2 n)q^n$ is in $\mathscr{E}(4p, \mathrm{id})$.

By Berkovich and Jagy, $TG_{1,p}$ consists of all the ternary quadratic forms with discriminant p^2 and $TG_{2,p}$ consists of the set of ternary quadratic forms with discriminant $16p^2$ and particular genus symbols. Then by Proposition 5 of [Lehman], both of the levels of ternary forms in $TG_{1,p}$ and $TG_{2,p}$ are $4p$ which implies that the theta series of $TG_{1,p}$ and $TG_{2,p}$ are all in $\mathscr{E}(4p, \mathrm{id})$.

Let

$$\theta_i(z) = \sum_{n=0}^{\infty} \left(\sum_{Q \in TG_{i,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|} \right) q^n, \ \ i = 1, \ 2.$$

Then $\theta(z), \ \theta_1(z), \ \theta_2(z)$ are all in $\mathscr{E}(4p, \mathrm{id})$.

Theorem

The theta series $\theta(z), \ \theta_1(z), \ \theta_2(z)$ are linearly independent in $\mathscr{E}(4p, \mathrm{id})$ and

$$\psi_p(z) = p\theta(z) + 48\theta_1(z) - 96\theta_2(z).$$

proof of the above Theorem

If $p \equiv 3 \pmod 4$, then considering the coefficients of the constant term, q-term and the q^4-term, we have

$$\begin{aligned}\theta(z) &= 1 + 6\cdot q + *\cdot q^2 + *\cdot q^3 + 6\cdot q^4 + \cdots,\\ \theta_1(z) &= \frac{p-1}{48} + \frac{1}{4}\cdot q + *\cdot q^2 + *\cdot q^3 + \frac{3}{4}\cdot q^4 + \cdots,\\ \theta_2(z) &= \frac{p-1}{48} + 0\cdot q + *\cdot q^2 + *\cdot q^3 + \frac{1}{4}\cdot q^4 + \cdots,\end{aligned}$$

where the symbol "*" means that we don't need to consider this coefficient. Since the matrix

$$A = \begin{pmatrix} 1 & (p-1)/48 & (p-1)/48 \\ 6 & 1/4 & 0 \\ 6 & 3/4 & 1/4 \end{pmatrix}$$

is invertible, $\theta(z),\ \theta_1(z),\ \theta_2(z)$ are linearly independent. Hence

$$\begin{aligned}\psi_p(z) &= 1 + 6(p+2)\cdot q + *\cdot q^2 + *\cdot q^3 + 6(p+2)\cdot q^4 + \cdots\\ &= p\theta(z) + 48\theta_1(z) - 96\theta_2(z).\end{aligned}$$

If $p \equiv 1 \pmod 4$, then by Lemma 2.3, we have

$$\begin{aligned}
\theta(z) &= 1 + 6 \cdot q + \cdots + S(p) \cdot q^p + \cdots \ , \\
\theta_1(z) &= \frac{p-1}{48} + \frac{1}{4} \cdot q + \cdots + \frac{1}{4} h(-p) \cdot q^p + \cdots \ , \\
\theta_2(z) &= \frac{p-1}{48} + 0 \cdot q + \cdots + 0 \cdot q^p + \cdots \ .
\end{aligned}$$

Since the matrix

$$A = \begin{pmatrix} 1 & \frac{p-1}{48} & \frac{p-1}{48} \\ 6 & 0 & 0 \\ S(p) & \frac{1}{4} h(-p) & 0 \end{pmatrix}$$

is invertible, $\theta(z)$, $\theta_1(z)$, $\theta_2(z)$ are linearly independent. And by Lemma 2.4, we have

$$\begin{aligned}
\psi_p(z) &= 1 + 6(p+2) \cdot q + \cdots + S(p^3) \cdot q^p + \cdots \\
&= p\theta(z) + 48\theta_1(z) - 96\theta_2(z) \ .
\end{aligned}$$

For those cases in Table 1 that are not covered in Table 3, it is also potential to derive the corresponding formulas for representation numbers through the similar arguments. The idea goes as follows. Let Q be a positive definite ternary form with integer coefficients. Then the associated modular form $\theta_Q(z)$ has weight $3/2$ and level N. If Q has class number 1, then $\theta_Q(z)$ lies in the space $\mathscr{E}(N, \omega)$. Note that Pei has constructed an explicit basis for $\mathscr{E}(N, \omega)$, so it is possible to write $\theta_Q(z)$ explicitly as a linear combination of this basis—-generally speaking this can be done by computing the values of $\theta_Q(z)$ at all its cusps. Here we simply benefit from Pei's computations for cases in Table 3, which was carried out in Pei's paper. For other cases in Table 1, we believe Cooper and Lam's conjecture holds too, although we have not been able to provide a rigorous proof.

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Thanks!