Jacobi forms and differential operators

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Introduction

$J_{k,m}(N)$ – The space of Jacobi forms of weight k, index m for the Jacobi group $\Gamma_0(N) \ltimes \mathbb{Z}^2$.
If ϕ is a Jacobi form, then $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is a holomorphic function satisfying the following properties:

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau, z), \qquad (1)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$\phi(\tau, z+\lambda\tau+\mu) = e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z),$$

for all $(\lambda, \mu) \in \mathbb{Z}^2$, and $\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2 \leq 4nm}} c(n,r)e^{2\pi i(n\tau+rz)}$, plus similar expansions at the cusps. We denote $J_{k,m}$ when $N = 1$.

Taylor Expansion: Consider the Taylor expansion of $\phi \in J_{k,m}(N)$ around $z = 0$:

$$\phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau) z^\nu.$$

Here χ_ν are the Taylor coefficients. The differential operators on Jacobi forms using these χ_ν are as follows.

$$D_\nu(\phi)(\tau) = A_{k,\nu}\xi_\nu = A_{k,\nu} \sum_{\mu=0}^{\nu/2} \frac{(-2\pi i m)^\mu (k+\nu-\mu-2)!}{(k+2\nu-2)!\mu!} \chi^{(\mu)}_{\nu-2\mu}(\tau),$$

where $A_{k,\nu} = (2\pi i)^{-\nu} \frac{(k+2\nu-2)!}{(k+\nu-2)!}(2\nu)!$. When $\nu = 0$, $\xi_0 = \chi_0$ and $\nu = 2$ gives $\xi_2 = \chi_2 - \frac{2\pi i m}{k}\chi_0^{(1)}$.

Theorem (Eichler-Zagier)
1. ξ_ν is a modular form of weight $k+\nu$ for $\Gamma_0(N)$. It is a cusp form if $\nu > 0$. $D_\nu : J_{k,m}(N) \to M_{k+2\nu}(N)$.
2. For all $m, N \geq 1$,
$\oplus_{\nu=0}^{m} D_{2\nu} : J_{k,m}(N) \hookrightarrow M_k(N) \oplus_{\nu=1}^{m} S_{k+2\nu}(N)$.
Note: D_0 is nothing but the restriction map $\phi(\tau, z) \mapsto \phi(\tau, 0)$, which maps the space $J_{k,m}(N)$ into $M_k(N)$.
The study of $\mathrm{Ker} D_0$ is useful in some applications.
(For example, a conjecture of Hashimoto on theta series attached to definite quaternion algebras).
It is known that D_0 is not injective when $k \geq 4$. So, the only interesting case occurs when $k = 2$. One of our main results deals with $k = 2$. An explicit description of $\mathrm{Ker} D_0$ ($m = 1, N =$ prime) was first studied by J. Kramer (1986).

Works of Arakawa-Böcherer

First we state two results proved by T. Arakawa and S. Böcherer (1999, 2003). We have $D_0 : J_{k,m}(N) \longrightarrow M_k(N)$. Set $J^0_{k,m}(N) = \mathrm{Ker} D_0$. Also set $S^0_{k+2}(N)$ as the image of $J^0_{k,m}(N)$ under the operator D_2.

Theorem(A-B) 1999

$J^0_{k,1}(N)$ is isomorphic to both $M_{k-1}(N, \omega)$ and $S^0_{k+2}(N)$. Moreover, the above two isomorphisms induce the following isomorphism given by

$$M_{k-1}(N, \omega) \simeq S^0_{k+2}(N),$$

$f \mapsto f\xi$, ξ = const. η^6 (a weight 3 modular form with character $\overline{\omega}$), where $\eta(\tau)$ is the Dedekind eta-function and ω is a character of $SL_2(\mathbb{Z})$.

The above fact suggests that one should study the subspace of $S_k(N)$ consisting of forms which are divisible by certain eta powers. This is what exactly they prove in their work.

Theorem (A-B) 2003

Let $S_k^{\eta}(N) := \{f \in S_k(N) : f \text{ is divisible by } \eta^{2k-2}(\tau)\}$. Then $S_k^{\eta}(N) = \{0\}$, when N is square-free, provided $k \equiv 4, 10 \pmod{12}$. (The same is true for $N = 1$ when k is not divisible by 12.)

Corollary When $k = 2$ and N is square-free, $S_4^0(N) = \{0\}$. Hence, $J_{2,1}^0(N) = \{0\}$. Therefore, D_0 is injective on $J_{2,1}(N)$, when N is square-free.

Thus, the works of Arakawa and Böcherer show that when $k = 2$, $m = 1$, one can remove the last differential operator (D_2) in the direct sum $D_0 \oplus D_2$ and still get the injectivity. The question is whether one can expect similar thing for $m \geq 2$.

Conjecture

The following conjecture is due to Böcherer (private communication):

Conjecture: For $k = 2$, N square-free, the direct sum $D_0 \oplus D_2 \oplus \ldots \oplus D_{2m-2}$ is injective on $J_{2,m}(N)$, when $m \geq 2$.

One can formulate a more general question: For $0 \leq \nu \leq m$, let $I_{2\nu}(k, m, N)$ be the property that the following map is injective:

$$D_0 \oplus \ldots \oplus \widehat{D_{2\nu}} \oplus \ldots \oplus D_{2m} :$$

$$J_{k,m}(N) \longrightarrow M_k(N) \oplus \ldots \oplus \widehat{M_{k+2\nu}(N)} \oplus \ldots \oplus M_{k+2m}(N).$$

Here $\widehat{}$ denotes the corresponding term in the direct sum to be omitted.

General Conjecture: $I_{2\nu}(2, m, N)$ is true for all $m \geq 1$, N square-free.

Results

We now state our results. First, we set
$I(k, m, N) := I_{2m}(k, m, N)$.

Theorem (with Karam Deo) 2013
$I(2, 2, N)$ is true for $N = 4$ or N is square-free.
i.e., $D_0 \oplus D_2$ is injective on $J_{2,2}(N)$ for the above values of N.

To prove this theorem, we followed the same procedure as in the work of Arakawa-Böcherer.

Our recent result (with Soumya Das) gives an affirmative answer in the more general case.

Theorem (with Soumya Das)
Let $k \geq 2$ be an even integer. Then
1. $I(k, m, N)$ is true for $k = m = 2$ and all square-free N,
2. $I(k, m, N)$ is true for $N \geq 1$ with $m - k \geq 1$,
3. $I(k, m, N)$ is true for N square-free and m odd with $m - k \geq -1$,
4. $I(k, m, N)$ is true for $N = 1$, $m - k \geq -1$

Corollary Conjecture 1 is true. i.e., $I(2, m, N)$ is true for $m \geq 1$ and N is square-free. Indeed,
from (3), we get $I(2, 1, N)$ is true.
(1) gives $I(2, 2, N)$ is true when N is square-free.
and from (2), we get $I(2, m, N)$ is true for $m \geq 3$.

Sketch

We now give a sketch of our proof.
Preliminary set up:
First consider the theta decomposition of a Jacobi form.

$$\phi(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\theta^J_{m,\mu}(\tau, z),$$

where $\theta^J_{m,\mu}(\tau, z) = \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} q^{r^2/4m}\zeta^r$, $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$ and

$$h_\mu(\tau) = h_{m,\mu}(\tau) = \sum_{\substack{n\in\mathbb{Z} \\ \mu^2/4m\leq n}} c_\phi(n, r)q^{(n-r^2/4m)}.$$

The theta (column) vector $Th_m(\tau, z) = (\theta^J_{m,\mu}(\tau, z))^t_{0\le\mu<2m}$ satisfies the transformation:

$$Th_m(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = e^{2\pi i m \frac{cz^2}{c\tau+d}}(c\tau + d)^{1/2} U_m(\gamma) Th_m(\tau, z),$$

where $U_m : SL_2(\mathbb{Z}) \to U(2m, \mathbb{C})$ is a projective representation of $SL_2(\mathbb{Z})$. The row vector $\mathbf{h}(\tau) = (h_{m,\mu}(\tau))_{0\le\mu<2m}$ satisfies the transformation

$$\mathbf{h}(\gamma\tau) = (c\tau + d)^{k-1/2}\mathbf{h}(\tau)\overline{U_m(\gamma)}^t$$

for all $\gamma \in \Gamma_0(N)$.

We have $\phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau) z^\nu$ and

$$\chi_{2\nu}(\tau) = \frac{1}{(2\nu)!} \phi^{(2\nu)}(\tau, z)|_{z=0}$$

Using the theta decomposition, this leads to

$$\chi_{2\nu}(\tau) = c_{\nu,m} \sum_{\mu=0}^{m} h_\mu \theta_{m,\mu}^{(\nu)},$$

where $\theta_{m,\mu}^{(\nu)}(\tau) = \theta_{m,\mu}^{(\nu)}(\tau, 0)$.

Thus we get the following.

$$\begin{pmatrix} \theta_{m,0} & \theta_{m,1} & \cdots & \theta_{m,m} \\ \theta^{(1)}_{m,0} & \theta^{(1)}_{m,1} & \cdots & \theta^{(1)}_{m,m} \\ \vdots & \vdots & \cdots & \vdots \\ \theta^{(m)}_{m,0} & \theta^{(m)}_{m,1} & \cdots & \theta^{(m)}_{m,m} \end{pmatrix} \begin{pmatrix} h'_0 \\ h'_1 \\ \vdots \\ h'_m \end{pmatrix} = \begin{pmatrix} \chi_0 \\ \chi_2 \\ \vdots \\ \chi_{2m} \end{pmatrix}$$

Here $h'_\mu = c_{\nu,m} h_\mu$. Let us denote the $(m+1) \times (m+1)$ matrix of the theta derivatives by $\mathcal{W}_m$ and let $W_m = \det \mathcal{W}_m$.

$W_m = const.\eta^{(m+1)(2m+1)}$ (J. Kramer)

Let $\phi \in \cap_{\nu=0}^{m-1} \mathrm{Ker} D_{2\nu}$. This implies from the description of $D_{2\nu}$ in terms of $\chi_{2\mu}$'s that

$$\phi \in \cap_{\nu=0}^{m-1} \mathrm{Ker} D_{2\nu} \iff \chi_{2\nu} = 0, \nu = 0, 1, \ldots, m-1.$$

So, for such a $\phi \in J_{k,m}(N)$, we get

$$h_\mu = c_m \chi_{2m} \frac{\omega_\mu}{W_m}, 0 \leq \mu \leq m,$$

where $c_m = c_{m,m}^{-1}$ and $(\omega_0/W_m, \cdots, \omega_m/W_m)^t$ is the last column of the matrix $\mathcal{W}_m^{-1}$.

First let us consider the case $k = m = 2$. If ϕ is non-zero, then we get $\chi_4 \neq 0$. Here χ_4 has weight 6 and thus we need a weight 10 cusp form in $\Gamma_0(N)$. Multiply χ_4 by a non-zero modular form $g \in M_4(N)$ and consider

$$F = \frac{\chi_4 g}{\eta^{18}}.$$

If we show that F is modular, then by virtue of the result of Arakawa-Bocherer we get $\chi_4 g = 0$, a contradiction and we are done.

Let N be square-free. The Fourier expansion of χ_4 at any cusp of $\Gamma_0(N)$ has the form

$$\chi_4|W(l) = \sum_{n\geq 1} a(n)q^n.$$

This is because we can choose $W(l)$ to be the Atkin-Lehner W operator (which normalizes $\Gamma_0(N)$), and hence $\chi_4|W(l)$ is again modular. Now $\eta^{18}(\tau)$ has the expansion

$$\alpha q^{18/24} + \text{higher terms}, \alpha \neq 0.$$

Thus, F is bounded at all the cusps and hence is modular. This method works only for $m = 2$.

Let $\Omega_m = (\omega_0, \cdots, \omega_m)^t$.
Claim: Ω_m is a vector-valued modular form on $SL_2(\mathbb{Z})$ with some multiplier v_m of weight $k = (2m^2 - m)/2$. We use the following fact to prove this. (This version is due to C. Marks).

Suppose κ is a real number and v is a multiplier system for the weight κ on $SL_2(\mathbb{Z})$. Then if V is a finite dimensional vector space with a right action $|_\kappa^v$ of $SL_2(\mathbb{Z})$, then to any set $\{f_1, \cdots, f_n\}$ which spans V, there is a representation $\rho : SL_2(\mathbb{Z}) \to GL(n, \mathbb{C})$ such that $(f_1, \cdots, f_n)^t$ is a vector valued modular form of weight κ and multiplier system v on $SL_2(\mathbb{Z})$.

We also use the following lemma. To prove it we use the Vandermonde determinant and Leibnitz expansion of determinants

Lemma.

$$\mathrm{ord}_{\infty}(\omega_m) = \frac{(m-1)(2m-1)}{24}.$$

For $\mu \neq 0$,

$$\mathrm{ord}_{\infty}(\omega_\mu) > \frac{(m-1)(2m-1)}{24}.$$

We have

$$h_\mu = c_m \chi_{2m} \frac{\omega_\mu}{W_m}$$

and $W_m = \det \mathcal{W}_m = const.\, \eta^\lambda$, $\lambda = (m+1)(2m+1)$. Multiply the above by a non-zero modular form $H \in M_s(N)$, $s \geq 0$. Let $\beta = 2(2m+k) + 2s - 2$ and $\alpha = (m-1)(2m-1)$. So,

$$const. \frac{\chi_{2m} H}{\eta^\beta} = \frac{h_\mu H}{\omega_\mu \eta^{\beta-\lambda}}.$$

We call the RHS function (for all μ such that $\omega_\mu \neq 0$) as ψ. So,

$$\psi = \frac{h_\mu H}{\omega_\mu \eta^{\beta-\lambda}}.$$

Next we put

$$\varphi(\tau) = \eta^{-r}\psi = \frac{h_\mu H}{\omega_\mu / \eta^{\lambda-\beta-r}}.$$

Note that φ has weight $1 - r/2$. Now, Ω_m is a vector valued modular form together with H, $(h_0, \cdots, h_m)$ modular imply

$$\varphi|^{\epsilon}_{(1-r/2)}\gamma$$

is bounded for $\gamma \in SL_2(\mathbb{Z})$ (i.e bounded at the cusps) with ϵ some multiplier system.
To get this we also need

$$\Omega_m(\tau) \to (0, 0, \cdots, *) \text{ as } \mathrm{Im}\tau \to \infty.$$

therefore,

$$\mathrm{ord}_{\infty}\omega_m = \mathrm{ord}_{\infty}\eta^{\lambda-\beta-r} = (\lambda-\beta-r)/24.$$

$\Longrightarrow \quad \alpha+\beta+r=\lambda$. i.e.,

$$k+s+r/2=m+1.$$

We now make choices of r, s, N, m in order to get a contradiction to prove our theorem. Throughout $k \geq 2$ is an even integer.

For proving (2), let $N \geq 1$. Choose $r > 2$ and $s = 0$. i.e., H is a non-zero constant. these choices lead to the condition $m - k \geq 1$ and also get a contradiction as φ is modular of weight $1 - r/2 < 0$.

If N is square-free, choose $r = 0$, $s = m - k + 1 \geq 0$ (this implies that m is odd), H a non-zero form in $M_s(N)$. To apply the A-B theorem, we need $2m + k + s \equiv 4, 10 \pmod{12}$, which leads to $m \equiv 1, 3 \pmod 4$. In this case, we get the condition $m - k \geq -1$.

If $N = 1$, choose $r = 0$, $s = m - k + 1 \geq 0$, H any non-zero modular form. To use the A-B theorem in this case the weight should not be a multiple of 12 and this leads to m any integer. So, we get the condition $m - k \geq -1$ for any integer m.

Some remarks

We make some remarks about the question of removing any differential operator.
For example, let $m = 1$, $N = 3$, In this case, the space $J_{2,1}(3)$ has non-zero elements, since $J_{2,1}(3)$ is isomorphic to the space $M^{+}_{3/2}(\Gamma_0(12))$ and $\dim M^{+}_{3/2}(\Gamma_0(12)) = 1$. However, $D_2 : J_{2,1}(3) \to S_4(3) = \{0\}$.
For $N > 3$, we don't know any example using dimension consideration.
So, the question in the general case (about removing any of the differential operator) is still open. Our method may not work in this case.

Another interesting case is when k is odd. In this case possibly one should consider the differential operators $D\nu$, where ν is odd.

Finally, what about the general congruence subgroup? It may be possible to carry out but the conditions on m and k may not be neat and it may depend on the group.

Thank You!!!