Lattices with many Borcherds products

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joint work with Jan Bruinier (Darmstadt)
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Explicit theory of Automorphic forms

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Finite quadratic modules

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- The *level* of $\mathcal{M}$ is the smallest positive integer $N \in \mathbb{Z}_{>0}$, such that $N \cdot Q(\mu) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $\mu \in M$.
- The signature $\operatorname{sig}(\mathcal{M}) \pmod 8$ is defined via

$$\frac{1}{\sqrt{|M|}} \sum_{\mu \in M} e(Q(\mu)) = e\left(\frac{\operatorname{sig}(\mathcal{M})}{8}\right).$$

Here, $e(x) = e^{2\pi i x}$.

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- Then the pair $\mathcal{M}_L = (L'/L, Q \pmod{\mathbb{Z}})$ is a finite quadratic module.
- Every fqm can be obtained this way.
- If the signature of L is (b^+, b^-), we have by Milgram's formula that $\operatorname{sig}(\mathcal{M}_L) \equiv b^+ - b^- \pmod{8}$.

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- and $M^!_{k,\mathcal{M}}$: weakly holomorphic modular forms ($c_f(n, \mu) = 0$ for $n < n_0 \in \mathbb{Z}$).

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- The weight of $\Psi(z, f)$ is $c_f(0, 0)/2$
- and we have
$$\operatorname{div}(f) = \sum_{\mu \in L'/L} \sum_{n<0} c_f(n, \mu) H(n, \mu),$$
for $H(n, \mu)$ the Heegner divisor of index (n, μ).

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- Obstructions for the existence of such an f:
  - $c(n,-\mu) = c(n,\mu)$,
  - for every $g \in S_{1+n/2,\mathcal{M}_L}$, we have

$$\sum_{\mu \in L'/L} \sum_{n<0} c_f(n,\mu) c_g(-n,\mu) = 0.$$

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- We call a lattice of signature $(2,n)$ *simple* if $S_{1+n/2,\mathcal{M}_L} = \{0\}$.
- Goal: Classify all isomorphism classes of simple lattices.

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- If $2k \not\equiv \operatorname{sig}(\mathcal{M}) \pmod{2}$, then $M_{k,\mathcal{M}} = \{0\}$.
- Assume that $2k \equiv -\operatorname{sig}(\mathcal{M}) \pmod{4}$. Let $W = \operatorname{span}\{\mathfrak{e}_\mu + \mathfrak{e}_{-\mu}\}$ and $d = \dim W$. Denote by ρ the restriction of $\rho_{\mathcal{M}}$ to W.

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- For a unitary matrix $A \in \mathbb{C}^{d \times d}$ with eigenvalues $e(\nu_j)$ for $j = 1, \ldots, d$ and $0 \leq \nu_j < 1$, we write $\alpha(A) = \nu_1 + \ldots + \nu_d$.

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Theorem

- *We have*

$$\dim M_{k,\mathcal{M}} = d + \frac{dk}{12} - \alpha\left(e(k/4)\rho(S)\right) - \alpha\left(\left(e(k/6)\rho(ST)\right)^{-1}\right) - \alpha(T)$$

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- $\dim S_{k,\mathcal{M}} = \dim M_{k,\mathcal{M}} - |\{\mu \in M/\{\pm 1\} \mid Q(\mu) \in \mathbb{Z}\}|$

$$+ \begin{cases} 0, & \textit{if } k \neq 2, \\ \dim M_{0,\mathcal{M}(-1)}, & \textit{if } k = 2. \end{cases}$$

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- We have

$$\alpha_3 + \alpha_4 \leq d.$$

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Remark

This bound is sharp! (We have $S_{14}(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$.)

Better estimates

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Let N be the level of $\mathcal{M}$. For $s \in \mathbb{R}$ define the divisor sum

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- $|\alpha_3 - d/2 + \alpha_4/2| \leq \frac{|M[2]|}{8} + \alpha_5/2$
- $$\alpha_5 \leq \frac{\sqrt{|M|}}{\pi}(3/2 + \ln(N))\left(\sigma(-1, \mathcal{M}) - \frac{\sqrt{|M|}}{N}\right)$$

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For every $\varepsilon > 0$ there is a $C > 0$, such that

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Corollary

Let $r \in \mathbb{Z}_{>0}$. There exist only finitely many isomorphism classes of finite quadratic modules $\mathcal{M}$ with minimal number of generators r, such that $S_{k,\mathcal{M}} = \{0\}$ for some $k \geq 2$ with $k \equiv -\operatorname{sig}(\mathcal{M}) \pmod 4$.

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- Problem for our application: if we make C explicit, the bounds on N we can obtain are huge.

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- We have $r = 2n+1$, $N = 3$, $\mathrm{sig}(M) = 6$ and

$$S_{3,\mathcal{M}} = \{0\}.$$

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- and an explicit formula for $\dim M_{0,\mathcal{M}(-1)}$!
- We did not try to do this.

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A finite quadratic module $\mathcal{M} = (M, Q)$ is called *isotropic* if there exists a $m \in M$, $m \neq 0$ with $Q(m) = 0 \in \mathbb{Q}/\mathbb{Z}$. Otherwise, $\mathcal{M}$ is called *anisotropic*.

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A finite quadratic module $\mathcal{M} = (M, Q)$ is called *isotropic* if there exists a $m \in M$, $m \neq 0$ with $Q(m) = 0 \in \mathbb{Q}/\mathbb{Z}$. Otherwise, $\mathcal{M}$ is called *anisotropic*.

Example

The anisotropic fqm's of odd prime level p are

$$\left(\mathbb{Z}/p\mathbb{Z}, \frac{ax^2}{p}\right), \quad a \in (\mathbb{Z}/p\mathbb{Z})^\times$$

and

$$\left((\mathbb{Z}/p\mathbb{Z})^2, \frac{x^2 - ay^2}{p}\right), \quad \left(\frac{a}{p}\right) = -1.$$

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- If $\mathcal{M}$ is anisotropic, $N = 2^t N'$, $t \in \{0, 1, 2, 3\}$ and N' is odd and squarefree.

An estimate for anisotropic modules

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Then

$$\alpha_5 = -\sum_{\substack{d \mid N \\ d(d,R)}} \epsilon(d)(N/d, R)H(-d(d, R)),$$

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Then

$$\alpha_5 = -\sum_{\substack{d \mid N \\ d(d,R)}} \epsilon(d)(N/d, R)H(-d(d, R)),$$

where $|\epsilon(d)| = 1$ and $H(-D)$ is the number of primitive positive definite quadratic forms of discriminant $-D$ for $D > 4$, $H(-3) = 1/3$, $H(-4) = 1/4$.

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Let $\mathcal{M}$ be an anisotropic fqm. Then

$$\dim S_{k,\mathcal{M}} \geq \frac{(|A|+1)(k-1)}{24} - 3 - 0.86\,|A|^{5/8}\ln(2\,|A|).$$

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Let $\mathcal{M}$ *be an anisotropic fqm. Then*

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Corollary

If $\mathcal{M} = (M, Q)$ *is anisotropic and* $2k \equiv -\operatorname{sig}(\mathcal{M}) \pmod{4}$ *with* $|M| \geq 5.3 \cdot 10^6$, *then* $S_{k,\mathcal{M}} \neq \{0\}$.

List of all k-simple anisotropic fqm's

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k	signature	k-simple finite quadratic modules	count
2	0	1^{+1}, 5^{-1}, $2_1^{+1}4_7^{+1}$, $3^{+1}11^{-1}$, $2^{-2}5^{+1}$, $2_2^{+2}3^{+1}$, $2_6^{+2}3^{-1}$, 13^{-1}, $2_6^{+2}7^{+1}$, 17^{+1}, $3^{-1}7^{-1}$, $2_1^{+1}4_1^{+1}3^{+1}$, $2_6^{+2}3^{+1}5^{+1}$	13
2	4	2^{-2}, 3^{+2}, 5^{+1}, 5^{-2}, $2_1^{+1}4_3^{-1}$, $3^{-1}11^{-1}$, $2^{-2}5^{-1}$, $2_2^{+2}3^{-1}$, $2_6^{+2}3^{+1}$, 13^{+1}, $2_2^{+2}7^{+1}$, 17^{-1}, $3^{+1}7^{-1}$, $2_1^{+1}4_1^{+1}3^{-1}$, $2_2^{+2}3^{-1}5^{-1}$	15
$\frac{5}{2}$	3	2_3^{+3}, 4_3^{-1}, $4_3^{-1}5^{-1}$, $2_1^{+1}3^{-1}$, $2_7^{+1}3^{+2}$, $2_1^{+1}7^{+1}$, $2_1^{+1}11^{-1}$, $4_1^{+1}7^{+1}$, $2_7^{+1}5^{+1}$, $4_1^{+1}3^{-1}$, $4_5^{-1}3^{+1}$, $2_1^{+1}3^{-1}5^{-1}$	12
$\frac{5}{2}$	7	2_7^{+1}, 4_7^{+1}, $2_1^{+1}3^{+1}$	3

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k	signature	k-simple finite quadratic modules	count
3	2	3^{-1}, 2_2^{+2}, $2^{-2}3^{+1}$, 7^{+1}, $2_1^{+1}4_1^{+1}$, 11^{-1}, $3^{+1}5^{+1}$, $3^{-1}5^{-1}$, $2_2^{+2}5^{-1}$, 23^{+1}	10
3	6	3^{+1}	1
$\frac{7}{2}$	1	2_1^{+1}, 4_1^{+1}, $2_7^{+1}3^{-1}$, $2_1^{+1}5^{-1}$, $4_3^{-1}3^{+1}$	5
$\frac{7}{2}$	5	4_5^{-1}	1
4	0	1^{+1}, 5^{-1}	2
4	4	5^{+1}, 2^{-2}	2
$\frac{9}{2}$	3	4_3^{-1}, $2_1^{+1}3^{-1}$	2
$\frac{9}{2}$	7	2_7^{+1}	1

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5	2	$3^{-1}, 2_2^{+2}, 7^{+1}$	2
5	6	3^{+1}	1
$\frac{11}{2}$	1	$2_1^{+1}, 4_1^{+1}$	2
6	0	1^{+1}	1
7	2	3^{-1}	1
$\frac{15}{2}$	1	2_1^{+1}	1
8	0	1^{+1}	1
10	0	1^{+1}	1
14	0	1^{+1}	1

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- and 50 isomorphism classes of modules.

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- We have an inclusion (induction from isotropic subgroups)

$$S_{k,\mathcal{M}/U} \hookrightarrow S_{k,\mathcal{M}}.$$

- Therefore, if $S_{k,\mathcal{M}/U} \neq \{0\}$, then $S_{k,\mathcal{M}} \neq 0$.

Our algorithm

Definition

For a finite quadratic module $\mathcal{M}$ and a positive integer n, we define

$$B(\mathcal{M}, n) = \{\mathcal{N} \mid \exists\, U \subset \mathcal{N}, \mathcal{N}/U = \mathcal{M}, |U| = n^2\}.$$

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- Nice property:
$$B(\mathcal{M}, m \cdot n) = \bigcup_{\mathcal{N} \in B(\mathcal{M}, m)} B(\mathcal{N}, n).$$
- Moreover, we can get very good bounds on primes p, such that for all $\mathcal{N} \in B(\mathcal{M}, p)$: $\dim S_{k,\mathcal{N}} > 0$. (The largest one is 37 for $k = 2$)

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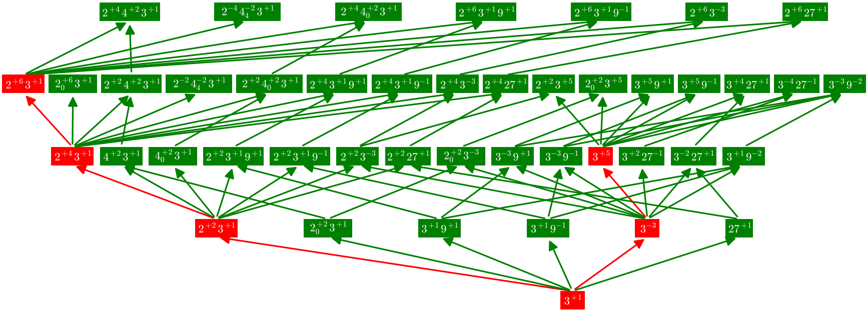
- Example: signature 6, $r = 6$, $k = 3$. (corresponds to signature $(2, 4)$)



Statistics

Statistics

- There are 90 finite quadratic modules of signature $2-n$ for $n \geq 2$ with minimal number of generators $\leq 2+n$, such that $S_{1+n/2,\mathcal{M}} = \{0\}$.

n	number	largest order
2	47	$625 : 5^{+4}$
3	16	$162 : 2_7^{+1}3^{+4}$
4	6	$243 : 3^{+5}$
5	4	$256 : 2^{+6}4_5^{-1}$
6	5	$256 : 2^{-8}$
7	2	$6 : 2_1^{+1}3^{-1}$
8	3	$7 : 7^{+1}$
9	3	$8 : 8^{+1}$
10	2	$4 : 2^{+2}$
18	1	1
26	1	1

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- Therefore, there are exactly 84 isomorphism classes of simple lattices of signature $(2, n)$ for $n \geq 2$.

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- Out of the 90 finite quadratic modules, 6 do not correspond to lattices of signature $(2, n)$.
- Since the discriminants are fairly low, the genera of all simple lattices each only contain a single isomorphism class.
- Therefore, there are exactly 84 isomorphism classes of simple lattices of signature $(2, n)$ for $n \geq 2$.
- Todo: $n = 1$.

Thank you for your attention.