Lattices with many Borcherds products
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joint work with Jan Bruinier (Darmstadt)
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Explicit theory of Automorphic forms

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Finite quadratic modules

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▶ The *level* of $\mathcal{M}$ is the smallest positive integer $N \in \mathbb{Z}_{>0}$, such that $N \cdot Q(\mu) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $\mu \in M$. 
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- We write \((\mu, \nu) = Q(\mu + \nu) - Q(\mu) - Q(\nu)\) for the associated bilinear form.
- The *level* of \( \mathcal{M} \) is the smallest positive integer \( N \in \mathbb{Z}_{>0} \), such that \( N \cdot Q(\mu) = 0 \in \mathbb{Q}/\mathbb{Z} \) for all \( \mu \in M \).
- The signature \( \text{sig}(\mathcal{M}) \pmod{8} \) is defined via

\[
\frac{1}{\sqrt{|M|}} \sum_{\mu \in M} e(Q(\mu)) = e\left(\frac{\text{sig}(\mathcal{M})}{8}\right).
\]

Here, \( e(x) = e^{2\pi ix} \).
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- Then the pair \(M_L = (L'/L, Q \pmod{\mathbb{Z}})\) is a finite quadratic module.
- Every fqm can be obtained this way.
- If the signature of \(L\) is \((b^+, b^-)\), we have by Milgram’s formula that \(\text{sig}(M_L) \equiv b^+ - b^- \pmod{8}\).
The Weil representation

Associated with $M$ is a representation $\rho_M$ of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[M]$.

$Mp_2(\mathbb{Z}) = \{ (A, \phi(\tau)) | A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \phi : \mathbb{H} \rightarrow \mathbb{C}, \phi^2(\tau) = c\tau + d \}$.

$Mp_2(\mathbb{Z})$ is generated by $S = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$, $T = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$.

We have $\rho_M(T)e^\mu = e^{-Q(\mu)}e^\mu \rho_M(S)e^\mu = e^{\frac{\text{sgn}(M)}{8}}\sqrt{|M|}\sum_{\nu \in M} e^{(\mu, \nu)}e^\nu$. 
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Vector valued modular forms

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$$f(\gamma \tau) = \phi(\tau)^{2k} \rho_{\mathcal{M}}((\gamma, \phi)) f(\tau).$$
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- and $M_{k,\mathcal{M}}^!$: weakly holomorphic modular forms ($c_f(n, \mu) = 0$ for $n < n_0 \in \mathbb{Z}$).
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- The weight of $\Psi(z, f)$ is $c_f(0, 0)/2$.
- and we have
  \[
  \text{div}(f) = \sum_{\mu \in L'/L} \sum_{n<0} c_f(n, \mu) H(n, \mu),
  \]
  for $H(n, \mu)$ the Heegner divisor of index $(n, \mu)$. 
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- The existence of a Borcherds product with a given divisor

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with \( c(n, \mu) \in \mathbb{Z} \), depends on the existence of \( f \in M_{1-n/2, M_L(-1)} \) with
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  \[ f = \sum_{\mu \in L'/L} \sum_{n < 0} c(n, \mu) e(n\tau) + \text{higher order terms}. \]

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- Obstructions for the existence of such an \( f \):
  - \( c(n, -\mu) = c(n, \mu) \),
  - for every \( g \in S_{1+n/2, M_L} \), we have

\[ \sum_{\mu \in L'/L} \sum_{n<0} c_f(n, \mu)c_g(-n, \mu) = 0. \]
Simple lattices

Therefore, if $S_{1+n/2}, \mathcal{M}_L(\mathbf{-1}) = \{0\}$, then every principal part with satisfying $c(n, -\mu) = c(n, \mu)$ occurs.

We call a lattice of signature $(2, n)$ simple if $S_{1+n/2}, \mathcal{M}_L = \{0\}$.

Goal: Classify all isomorphism classes of simple lattices.
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- Assume that \( 2k \equiv -\text{sig}(\mathcal{M}) \pmod{4} \). Let \( W = \text{span}\{e_{\mu} + e_{-\mu}\} \) and \( d = \dim W \). Denote by \( \rho \) the restriction of \( \rho_{\mathcal{M}} \) to \( W \).
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- For a unitary matrix $A \in \mathbb{C}^{d \times d}$ with eigenvalues $e(\nu_j)$ for $j = 1, \ldots, d$ and $0 \leq \nu_j < 1$, we write $\alpha(A) = \nu_1 + \ldots + \nu_d$. 
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Theorem

We have

$$\dim M_{k, \mathcal{M}} = d + \frac{dk}{12} - \alpha\left(e(k/4)\rho(S)\right) - \alpha\left((e(k/6)\rho(ST))^{-1}\right) - \alpha(T).$$
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- **dim $S_{k,\mathcal{M}} = \dim M_{k,\mathcal{M}} - |\{\mu \in M/\{\pm 1\} \mid Q(\mu) \in \mathbb{Z}\}|$$**

  $$
  + \begin{cases} 
  0, & \text{if } k \neq 2, \\
  \dim M_{0,\mathcal{M}(-1)}, & \text{if } k = 2.
  \end{cases}
  $$
Trivial estimates

Note that $d = |\mathcal{M}/\{\pm 1\}|$.

We have $\alpha_1 = \alpha(e(k/4)\rho(S)) \leq 1/2d$, $\alpha_2 = \alpha((e(k/6)\rho(ST)) - 1) \leq 2/3d$.

Let $\alpha_3 = \rho(T)$ and $\alpha_4 = |\{\mu \in \mathcal{M}/\{\pm 1\} | Q(\mu) \in \mathbb{Z}\}|$.

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Corollary

If \( k > 14 \) and \( 2k \equiv -\text{sig}(\mathcal{M}) \mod 4 \), then \( S_{k,\mathcal{M}} \neq \{0\} \).
Trivial estimates

Corollary

*If* $k > 14$ *and* $2k \equiv -\text{sig}(\mathcal{M}) \pmod{4}$, *then* $S_{k,\mathcal{M}} \neq \{0\}$.

Remark

This bound is sharp! (We have $S_{14}(\text{SL}_2(\mathbb{Z})) = \{0\}$.)
Better estimates

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Let $N$ be the level of $\mathcal{M}$. For $s \in \mathbb{R}$ define the divisor sum

$$\sigma(s, \mathcal{M}) = \sum_{a \mid N} a^s |M[a]|.$$
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\[
\sigma(s, \mathcal{M}) \leq \sqrt{\frac{2|\mathcal{M}|}{N}} \sigma_{s+1/2}(N).
\]
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\[ |\alpha_1 - d/4| \leq \frac{1}{4} \sqrt{|M[2]|}, \]

\[ |\alpha_2 - d/3| \leq \frac{1}{3} \sqrt{3 \left(1 + \sqrt{|M[3]|}\right)}, \]

\[ \alpha_3 \leq |M[2]|^2 + \sqrt{|M|} \sigma(-1, M), \]

\[ |\alpha_3 - d/2 + \alpha_4/2| \leq |M[2]|^8 + \alpha_5^2, \]

\[ \alpha_5 \leq \sqrt{|M|} \pi \left(\frac{3}{2} + \ln(N)\right) \left(\sigma(-1, M) - \sqrt{|M|}N\right). \]
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Proposition

*We have*

- \[ |\alpha_1 - d/4| \leq \frac{1}{4} \sqrt{|M[2]|}, \]
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- \(\alpha_4 \leq \frac{|M[2]|}{2} + \frac{\sqrt{|M|}}{2} \sigma(-1, \mathcal{M}),\)
- \(|\alpha_3 - d/2 + \alpha_4/2| \leq \frac{|M[2]|}{8} + \alpha_5/2\)
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- \(|\alpha_3 - d/2 + \alpha_4/2| \leq \frac{|M[2]|}{8} + \alpha_5/2\)
- \(\alpha_5 \leq \frac{\sqrt{|M|}}{\pi} (3/2 + \ln(N)) \left(\sigma(-1, \mathcal{M}) - \sqrt{|M|}/N\right)\)
Better estimates

Theorem (BEF)
For every \( \varepsilon > 0 \) there is a \( C > 0 \), such that
\[
\dim S_{k,M} \geq d(k - 1 + \frac{1}{12} - CN\varepsilon^{-1/2}).
\]

Corollary
Let \( r \in \mathbb{Z}_{>0} \). There exist only finitely many isomorphism classes of finite quadratic modules \( M \) with minimal number of generators \( r \), such that \( S_{k,M} = \{0\} \) for some \( k \geq 2 \) with \( k \equiv -\sigma(M) \pmod{4} \).

Problem for our application: if we make \( C \) explicit, the bounds on \( N \) we can obtain are huge.
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Let $r \in \mathbb{Z}_{>0}$. There exist only finitely many isomorphism classes of finite quadratic modules $\mathcal{M}$ with minimal number of generators $r$, such that $S_{k,\mathcal{M}} = \{0\}$ for some $k \geq 2$ with $k \equiv -\text{sig}(\mathcal{M}) \pmod{4}$. 

Problem for our application: if we make $C$ explicit, the bounds on $N$ we can obtain are huge.
Better estimates

**Theorem (BEF)**

For every $\varepsilon > 0$ there is a $C > 0$, such that

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- Consider the finite quadratic module $\mathcal{M}$ given by $M = (\mathbb{Z}/3\mathbb{Z})^{2n+1}$ and $Q(x) = \frac{1}{3}(x_1^2 + \ldots + x_{2n}^2 - (-1)^n x_{2n+1}^2)$. 
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- We have $r = 2n + 1$, $N = 3$, $\text{sig}(M) = 6$ and

$$S_{3,M} = \{0\}.$$
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- We did not try to do this.
Anisotropic modules

Definition

A finite quadratic module \( M = (M, Q) \) is called isotropic if there exists a \( m \in M, m \neq 0 \) with \( Q(m) = 0 \in Q/\mathbb{Z} \). Otherwise, \( M \) is called anisotropic.

Example

The anisotropic fqm's of odd prime level \( p \) are \( (\mathbb{Z}/p\mathbb{Z}, ax^2p) \) and \( (\mathbb{Z}/p\mathbb{Z})^2, x^2 - ay^2p) \), \( (a^p) = -1 \).

If \( M \) is anisotropic, \( N = 2^t N' \), \( t \in \{0, 1, 2, 3\} \) and \( N' \) is odd and squarefree.
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Let $\mathcal{M} = (M, Q)$ be an anisotropic fqam. Assume for simplicity that $|M|$ is odd.
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Let $\mathcal{M} = (M, Q)$ be an anisotropic fqm. Assume for simplicity that $|M|$ is odd. Define

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Then

$$\alpha_5 = - \sum_{d \mid N \atop d(d, R)} \epsilon(d)(N/d, R) H(-d(d, R)),$$
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\[ \alpha_5 = - \sum_{\substack{d | N \quad d(d,R)}} \epsilon(d)(N/d, R) H(-d(d, R)), \]

where $|\epsilon(d)| = 1$ and $H(-D)$ is the number of primitive positive definite quadratic forms of discriminant $-D$ for $D > 4$, $H(-3) = 1/3$, $H(-4) = 1/4$. 
Estimates

Lemma
We have $H(−D) \leq \sqrt{D \ln D} \pi$.

Corollary
Let $M$ be an anisotropic fqm. Then $\dim S_k, M \geq (|A| + 1)(k - 1) - 3 - 0.86 \frac{5}{8} \ln(2|A|)$.

Corollary
If $M = (M, Q)$ is anisotropic and $2^k \equiv -\sigma_4(M) \pmod{4}$ with $|M| \geq 5.3 \cdot 10^6$, then $S_k, M \neq \{0\}$. 
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Let \( \mathcal{M} \) be an anisotropic fqm. Then

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\dim S_{k,\mathcal{M}} \geq \frac{(|A| + 1)(k - 1)}{24} - 3 - 0.86 |A|^{5/8} \ln(2 |A|).
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If \( \mathcal{M} = (M, Q) \) is anisotropic and \( 2k \equiv -\text{sig}(\mathcal{M}) \pmod{4} \) with \( |M| \geq 5.3 \cdot 10^6 \), then \( S_{k,\mathcal{M}} \neq \{0\} \).
List of all $k$-simple anisotropic fqms
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<table>
<thead>
<tr>
<th>$k$</th>
<th>signature</th>
<th>$k$-simple finite quadratic modules</th>
<th>count</th>
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<td>13</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$2^-2, 3^+2, 5^+1, 5^-2, 2^{+1}_14^-3, 3^-111^-1, 2^-25^-1,$ $2^{+2}3^-1, 2^{+2}3^+1, 13^+1, 2^{+2}7^+1, 17^-1,$ $3^+17^-1, 2^{+1}_14^+_13^-1, 2^{+2}3^-15^-1$</td>
<td>15</td>
</tr>
<tr>
<td>$\frac{5}{2}$</td>
<td>3</td>
<td>$2^{+3}_3, 4^-1, 4^-15^-1, 2^+_13^-1, 2^+_13^+2, 2^+_17^+1,$ $2^{+1}_111^-1, 4^+_17^+1, 2^+_715^+1, 4^+_13^-1, 4^-13^+1,$ $2^{+1}_13^-15^-1$</td>
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</tr>
<tr>
<td>$\frac{5}{2}$</td>
<td>7</td>
<td>$2^+_7, 4^+_7, 2^+_13^+1$</td>
<td>3</td>
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| 3   | 2         | $3^{-1}, 2_{2}^{+2}, 2^{-2}3^{+1}, 7^{+1}, 2_{1}^{+1}4_{1}^{+1}, 11^{-1},$  
|     |           | $3^{+1}5^{+1}, 3^{-1}5^{+1}, 2_{2}^{+2}5^{-1}, 23^{+1}$ | 10    |
| 3   | 6         | $3^{+1}$                           | 1     |
| $\frac{7}{2}$ | 1         | $2_{1}^{+1}, 4_{1}^{+1}, 2_{7}^{+1}3^{-1}, 2_{1}^{+1}5^{-1}, 4_{3}^{+1}3^{+1}$ | 5     |
| $\frac{7}{2}$ | 5         | $4_{5}^{-1}$                        | 1     |
| 4   | 0         | $1^{+1}, 5^{-1}$                    | 2     |
| 4   | 4         | $5^{+1}, 2^{-2}$                    | 2     |
| $\frac{9}{2}$ | 3         | $4_{3}^{-1}, 2_{1}^{+1}3^{-1}$      | 2     |
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## List of all $k$-simple anisotropic fqm's

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<tbody>
<tr>
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<td>$3^{-1}, 2^2, 7^1$</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>$3^1$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{11}{2}$</td>
<td>1</td>
<td>$2^1, 4^1$</td>
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<tr>
<td>6</td>
<td>0</td>
<td>$1^1$</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>$3^{-1}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{15}{2}$</td>
<td>1</td>
<td>$2^1$</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>$1^1$</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>$1^1$</td>
<td>1</td>
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<tr>
<td>14</td>
<td>0</td>
<td>$1^1$</td>
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In total 78 pairs \((M, k)\) and 50 isomorphism classes of modules.
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- Therefore, if $S_{k, \mathcal{M}/U} \neq \{0\}$, then $S_{k, \mathcal{M}} \neq 0.$
Our algorithm

Definition

For a finite quadratic module $\mathcal{M}$ and a positive integer $n$, we define

$$B(\mathcal{M}, n) = \{ \mathcal{N} \mid \exists U \subset \mathcal{N}, \mathcal{N}/U = \mathcal{M}, |U| = n^2 \}. $$
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▶ Nice property:

$$B(\mathcal{M}, m \cdot n) = \bigcup_{\mathcal{N} \in B(\mathcal{M}, m)} B(\mathcal{N}, n).$$

▶ Moreover, we can get very good bounds on primes $p$, such that for all $\mathcal{N} \in B(\mathcal{M}, p)$: $\dim S_{k, \mathcal{N}} > 0$. (The largest one is 37 for $k = 2$)
Our algorithm
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Example: signature $6, r = 6, k = 3$. (corresponds to signature $(2, 4)$)
Statistics

There are 90 finite quadratic modules of signature $2 - n$ for $n \geq 2$ with minimal number of generators $\leq 2 + n$, such that $S_1 + n/2, M = \{0\}$.
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<tr>
<td>2</td>
<td>47</td>
<td>$625 : 5^{+4}$</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>$162 : 2_7^{+1} 3^{+4}$</td>
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<td>4</td>
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<td>$243 : 3^{+5}$</td>
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<td>4</td>
<td>$256 : 2^{+6} 4^{-1}$</td>
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<td>6</td>
<td>5</td>
<td>$256 : 2^{-8}$</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>$6 : 2_1^{+1} 3^{-1}$</td>
</tr>
<tr>
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<td>3</td>
<td>$7 : 7^{+1}$</td>
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<td>9</td>
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<tr>
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<tr>
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Finally: Simple lattices

Out of the 90 finite quadratic modules, 6 do not correspond to lattices of signature (2, n).
Since the discriminants are fairly low, the genera of all simple lattices each only contain a single isomorphism class.
Therefore, there are exactly 84 isomorphism classes of simple lattices of signature (2, n) for n ≥ 2.

Todo: n = 1.
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Thank you for your attention.