Dimension formulas for vector-valued Hilbert modular forms

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Possible applications

- Jacobi forms over number fields
  - Same type of correspondence as over $\mathbb{Q}$ (between scalar and vector-valued)
  - Liftings between Hilbert modular forms and Jacobi forms (Shimura lift)
- Independent applications for the reduction algorithms:
  - Reduction of hyperelliptic curves

Preliminaries

- $K/\mathbb{Q}$ number field of degree n
- O_K the ring of integers of K.
- Embeddings: $\sigma_i : K \to \mathbb{R}$, $1 \leq i \leq n$,
- Trace and norm:

$$\mathrm{Tr}\alpha = \sum \sigma_i \alpha, \quad \mathrm{N}\alpha = \prod \sigma_i \alpha.$$

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(K)$ we write $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} \sigma_i(a) & \sigma_i(b) \\ \sigma_i(c) & \sigma_i(d) \end{pmatrix}$.

Generalised upper half plane

- The group $\mathrm{SL}_2(K) \subset \mathrm{M}_2(K)$ acts on

$$\mathbb{H}^n \simeq \mathbb{H} \times \cdots \times \mathbb{H} = \{(z_1, \ldots, z_n) \mid z_j \in \mathbb{H}\}$$

  by

$$Az = (A_1 z_1, \ldots, A_n z_n) \in \mathbb{H}^n$$

  where $A_i z_i$ is the usual action of $PSL_2(\mathbb{R})$ on the upper half-plane $\mathbb{H}$.
- The (full) Hilbert modular group is defined as:

$$\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}$$

- Important: the definition of “the” Hilbert modular group is not canonical and other choices exist.

Hilbert modular forms

- Let $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$
- If $f : \mathbb{H}^n \to \mathbb{C}$ is holomorphic and satisfies

$$f(Az) = J_A(z;k) f(z)$$

where $J_A(z;k) = \prod (c_i z_i + d_i)^{k_i}$ then we say that f is a Hilbert modular form on Γ_K of weight k.

Vector-valued Hilbert modular forms

- Let $\rho : \Gamma_K \to GL(r, \mathbb{C})$ be a finite dimensional representation of Γ_K s.t.
  - $\mathrm{Ker}(\rho) = \Gamma$ a finite index subgroup of Γ_K
  - If $\alpha \in Z(\Gamma_K)$ then
$$\rho(\alpha) J_\alpha(z;k) = 1_{r\times r} \qquad (*)$$
- If $f : \mathbb{H}^n \to \mathbb{C}^r$ is holomorphic and satisfies
$$f(Az) = J_A(z;k)\rho(A) f(z)$$
  for all $A \in \Gamma_K$ then f is said to be a vector-valued Hilbert modular form of weight k and representation ρ.
- Denote the space of these by $M_k(\rho)$
- Note that (*) implies that $f(\alpha z) = \rho(\alpha) J_\alpha(z;k) f(z) = f(z)$
- If $f \in M_k(\rho)$ and $f = \sum f_i v_i$ then $f_i \in M_k(\Gamma)$ (scalar-valued)
$$S_k(\rho) = \left\{ f = \sum f_i v_i \in M_k(\rho), : \ f_i \in S_k(\Gamma) \right\}$$

Main theorem

If $k \in \mathbb{Z}^n$ with $k \gg 2$ then:

$$\dim S_k(\rho) = \frac{1}{2}\dim\rho \cdot \zeta_K(-1) \cdot \mathrm{N}(k-1)$$
$$+\text{"elliptic terms"}$$
$$+\text{"parabolic terms}$$

- Identity (main) term: $\zeta_K(-1)$ (a rational number)
  - Example: $\zeta_{\mathbb{Q}(\sqrt{5})} = \frac{1}{30}$, $\zeta_{\mathbb{Q}(\sqrt{193})}(-1) = 16 + \frac{1}{3}$, $\zeta_{\mathbb{Q}(\sqrt{1009})}(-1) = 211$.
- Finite order (“elliptic”) terms
- Parabolic (“cuspidal”) term

Remark

We have also shown the corresponding theorem for half-integral weight.

The elliptic terms

$$\text{"elliptic terms"} = \sum_{\mathfrak{U}} \frac{1}{|\mathfrak{U}|} \sum_{\pm 1 \neq A \in \mathfrak{U}} \chi_\rho(A) \cdot E(A)$$

here $\mathfrak{U}$ runs through elliptic conjugacy classes and

$$\begin{aligned} \chi_\rho(A) &= \mathrm{Tr}\rho(A), \\ E(A) &= \prod_{i=1}^{n} \frac{r(A_i)^{1-k_\sigma}}{r(A_i) - r(A_i)^{-1}} \\ r(A) &= \frac{1}{2}\left(t + \operatorname{sgn}(c)\sqrt{t^2-4}\right), \; t = \mathrm{Tr}A \end{aligned}$$

Note that if $Az^* = z^*$ then $r(A) = cz^* + d = j_A(z^*)$.

Cusps of $\mathrm{SL}_2(\mathcal{O}_K)$

- Cusp: $\lambda = (\rho : \sigma) \in \mathbb{P}_1(K)$
- Fractional ideal: $\mathfrak{a}_\lambda = (\rho, \sigma)$
- $\lambda \sim \mu \pmod{\mathrm{SL}_2(\mathcal{O}_K)} \Leftrightarrow \mathfrak{a}_\lambda = (\alpha)\,\mathfrak{a}_\mu$
- The number of cusp classes equals the class number of K (we assume this is $= 1$).
- Cusp-normalizing map: $\exists \xi, \eta \in \mathfrak{a}_\lambda^{-1}$ s.t.

$$\begin{aligned} A_\lambda &= \begin{pmatrix} \rho & \xi \\ \sigma & \eta \end{pmatrix} \in \mathrm{SL}_2(K), \\ A_\lambda^{-1}\mathrm{SL}_2(\mathcal{O}_K)A_\lambda &= \mathrm{SL}_2\left(\mathfrak{a}^2 \oplus \mathcal{O}_K\right) \end{aligned}$$

Cuspidal term

Contribution of the cusp λ is the value at $s = 1$ of a twisted Shimizu L-series

$$L(s;\lambda,\rho) = \frac{\sqrt{|d_K|}\mathrm{N}(\mathfrak{a}_\lambda^{-2})}{(-2\pi i)^n} \sum_{0 \neq a \in \mathfrak{a}_\lambda^{-2}/U^2} \chi_{\overline{\rho}}\left(A_\lambda^{-1}\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)A_\lambda\right) \frac{\mathrm{sgn}(\mathrm{N}(a))}{|\mathrm{N}(a)|^s}.$$

The "untwisted" L-series ($\rho = 1$) is known to have analytic cont. and functional equation

$$\Lambda(s) = \Gamma\left(\frac{s+1}{2}\right)^n \left(\frac{\mathrm{vol}(\mathcal{O}_K)}{\pi^{n+1}}\right)^s L(s;\mathcal{O}_K,1) = \Lambda(1-s)$$

- It is easy to see that the L-function for $\rho \neq 1$ also has AC. FE is more complicated (cf. Hurwitz-Lerch).
- If K has a unit of norm -1 then $L(s;\mathcal{O}_K,1) = 0$ (conditions on ρ in general)

Notes on the L-series

- Note that $L(s; \mathcal{O}_K, 1)$ is proportional to

$$L(s,\chi) = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{|\mathrm{N}(\mathfrak{a})|^s}$$

  where the sum is over all integral ideals of $\mathcal{O}_K$ and $\chi(\mathfrak{a}) = \mathrm{sgn}(\mathrm{N}(\mathfrak{a}))$.
- Studied by Hecke, Siegel, Meyer, Hirzebruch and others.
- Can be expressed in terms of Dedekind sums (Siegel)

Example $\mathbb{Q}\left(\sqrt{3}\right)$

- By Siegel (see e.g. Gundlach): $L\left(1, \mathrm{sgn} \circ \mathrm{N}\right) = \frac{\pi^2}{12\sqrt{3}}$.
- Our parabolic term is then:

$$L\left(1; \infty, 1\right) = -\frac{1}{6}.$$

- Example $k_1 = k_2 = 2$: Scalar term

$$\frac{1}{2}\zeta_{\sqrt{3}}\left(-1\right) = \frac{1}{12}$$

  - Elliptic terms (there are 3 order 4 classes, 2 order 6 and 1 order 12):

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{9} + \frac{1}{9} + \frac{35}{72} = 1 + \frac{1}{12}$$

  - $\dim S_{2,2}\left(1\right) = 1$.
- Example $k_1 = k_2 = 4$:
  - Scalar term $= \frac{3}{4}$
  - Elliptic terms:

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} - \frac{2}{9} - \frac{2}{9} + \frac{35}{72} = \frac{5}{12}$$

  - $\dim S_{4,4}\left(1\right) = \frac{3}{4} + \frac{5}{12} - \frac{1}{6} = 1$.

Example

We can compute dimensions of congruence subgroups.

Let $K = \mathbb{Q}\left(\sqrt{5}\right)$, $\mathfrak{m} = \left(\sqrt{5}\right)$ and consider

$$\Gamma_0\left(\mathfrak{m}\right) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_K,\ c \in \mathfrak{m} \right\}$$

and let $\rho = \mathrm{Ind}_{\Gamma_0(\mathfrak{m})}^{\Gamma_K}$ be the induced representation. Then we can compute the dimensions:

$k_1 = k_2$	$\dim S_k\left(\Gamma_K\right)$	$\dim S_k\left(\Gamma_0\left(\mathfrak{m}\right)\right)$
2	1	1
4	0	0
6	1	3
8	1	5

Conjugacy classes

- Scalar if $A = \pm 1$
- Elliptic: A has finite order.
- Parabolic: If A is not scalar but $\mathrm{Tr} A = \pm 2$.
- Mixed (these do not contribute to the dimension formula).

Note: the names “elliptic” and “parabolic” are not standard for Hilbert modular groups.

There are two main computational tasks

1. Elliptic contribution:
   - The terms are easy to compute
   - The problem is to find the classes (representatives)
2. Cuspidal contribution:
   - The conjugacy classes are easy to find.
   - The problem is to compute $L(1; \mathcal{O}_K, \rho)$.

How do we find elliptic conjugacy classes?

- Characterisation / parametrisation of elliptic elements: $(t; x, y) \to z_{t,x,y}$
- This is an infinite list!
- Use a reduction algorithm for Γ_K to obtain a finite set of reduced points.
- Choice of fundamental domain for Γ_K.

Which orders can appear?

Lemma

If A in Γ_K has order m then $\varphi(m) = 2d$ where d divides $n = \deg K$.

If $K = \mathbb{Q}(\sqrt{D})$ then the possible orders are:

- $3, 4, 6$ (solutions of $\varphi(l) = 2$), and
- $5, 8, 10, 12$ (solutions of $\varphi(l) = 4$)

Parametrisation of elliptic elements

Lemma

Let $\mathfrak{a}$ be a fractional ideal and $t \in K$ be such that $|t| \ll 2$. Then

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \lambda(A) = \frac{a - d + \sqrt{t^2 - 4}}{2c}$$

is a bijection between the set of elements of $SL_2(\mathfrak{a} \oplus \mathcal{O}_K)$ with trace t and

$$\left\{ z_{t,x,y} = \frac{x + \sqrt{t^2 - 4}}{2y} \in \mathbb{H}_K : x \in \mathcal{O}_K, y \in \mathfrak{a}, \; x^2 - t^2 + 4 \in 4\mathcal{O}_K \right\}.$$

To compute elliptic classes

- We choose a (closed) fundamental domain $\mathcal{F}_K$ of Γ_K.
- There are EXPLICIT bounds on x, y for $z_{t,x,y} \in \mathcal{F}_K \to$ finite list.
- Note that there are formulas for the number of elliptic elements (for quadratic K) but we need to know the actual matrices.
- Main problem:
  - How do we know whether two reduced elliptic points (in the fundamental domain) are equivalent or not?
  - The identifying matrix can be complicated.
  - IDEALLY: follow the "bottom" of fundamental domain to get generators and relations.

Distance to a cusp

- Distance to infinity

$$\Delta(z,\infty) = \mathrm{N}(y)^{-\frac{1}{2}}$$

- Distance to other cusps

$$\Delta(z,\lambda) = \Delta\left(A_\lambda^{-1}z,\infty\right).$$

- λ is a closest cusp to z if

$$\Delta(z,\lambda) \leq \Delta(z,\mu), \quad \forall \mu \in \mathbb{P}^1(K).$$

Lattices related to K

- $\mathcal{O}_K$ the ring of integers with integral basis $1 = \alpha_1, \alpha_2, \ldots \alpha_n$

$$\mathcal{O}_K \simeq \alpha_1\mathbb{Z} \oplus \cdots \oplus \alpha_n\mathbb{Z},$$

- $\mathcal{O}_K^\times$ the unit group with generators $\pm 1, \varepsilon_1, \ldots, \varepsilon_{n-1}$

$$\mathcal{O}_K^\times \quad \simeq \quad \langle \pm 1 \rangle \times \langle \varepsilon_1 \rangle \times \cdots \langle \varepsilon_{n-1} \rangle$$

- Λ the logarithmic unit lattice: $v_i = (\ln|\sigma_1\varepsilon_i|, \ldots, \ln|\sigma_{n-1}\varepsilon_i|)$

$$\Lambda = v_1\mathbb{Z} \oplus \cdots \oplus v_{n-1}\mathbb{Z}.$$

  The volume of Λ is called the regulator $\operatorname{Reg}(K)$.
- The volume of $\mathcal{O}_K$ is $|d_K|^{\frac{1}{2}}$, d_K is the discriminant of K.
- We denote Gram matrices of the above lattices by $B_{\mathcal{O}_K}$ and Λ.

Example $\mathbb{Q}\left(\sqrt{5}\right)$

In $\mathbb{Q}\left(\sqrt{5}\right)$ we have the fundamental unit ε and its conjugate ε^*:

$$\varepsilon_0 = \frac{1}{2}\left(1+\sqrt{5}\right), \quad \varepsilon^* = -\varepsilon_0^{-1} = \frac{1}{2}\left(1-\sqrt{5}\right).$$

And

$$\begin{aligned} \mathcal{O}_K &\simeq \mathbb{Z}+\varepsilon_0\mathbb{Z}, \\ \Lambda &\simeq \mathbb{Z}\ln\left|\frac{1+\sqrt{5}}{2}\right| \end{aligned}$$

with the volume given by

$$\begin{aligned} |\mathcal{O}_K| &= \left|\det\begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & \frac{1}{2}\left(1-\sqrt{5}\right) \\ 1 & 1 \end{pmatrix}\right| = \sqrt{5} \\ |\Lambda| &= \left|\ln\frac{1}{2}\left(1+\sqrt{5}\right)\right| \simeq 0.4812\ldots \end{aligned}$$

Generators

It is known that $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by (for example)

$$\begin{aligned} T^{\alpha} &= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \ \alpha = \alpha_1, \dots, \alpha_n, \\ S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ E(\varepsilon) &= \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \varepsilon = -1, \varepsilon_1, \dots, \varepsilon_{n-1} \end{aligned}$$

Problem: can not generalise Poincaré from $\mathrm{SL}_2(\mathbb{R})$. That is we can not obtain a fundamental domain with sides identified by the above generators.

Reduction algorithm for $z \in \mathbb{H}_K$

- For simplicity assume class number one.
- Find closest cusp λ and set $z^* = x^* + iy^* = A_\lambda^{-1} z$ (∞ is closest cusp of z^*).
- z^* is $\mathrm{SL}_2(\mathcal{O}_K)$ - reduced if it is $\Gamma_{K,\infty}$- reduced, where

$$\Gamma_{K,\infty} = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix}, \varepsilon \in \mathcal{O}_K^\times, \mu \in \mathcal{O}_K \right\}.$$

- Local coordinate (w.r.t.. lattices Λ and $\mathcal{O}_K$):

$$\begin{aligned} \Lambda Y &= \tilde{y} \\ B_{\mathcal{O}_K} X &= x^* \end{aligned}$$

where $Y \in \mathbb{R}^{n-1}$, $X \in \mathbb{R}^n$ and $\tilde{y}_i = \ln \frac{y_i^*}{\sqrt[n]{\mathrm{N} y^*}}$.

Reduction algorithm in cuspidal nbhd

- Then z^* is $\Gamma_{K,\infty}$- reduced iff

$$X_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], \; 1 \le i \le n,$$
$$Y_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], \; 1 \le i \le n-1.$$

- If z is not $\Gamma_{K,\infty}$ reduced we can reduce:
  - Y by acting with $\varepsilon = \varepsilon_1^{m_1} \cdots \varepsilon_n^{m_n} \in \mathcal{O}_K^\times$:

$$U(\varepsilon) = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) : z^* \mapsto \varepsilon^2 z^*, \; Y_i \mapsto Y_i + m_i.$$

  - X by acting with $\zeta = \sum a_i \alpha_i \in \mathcal{O}_K$:

$$T(\zeta) = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} : z^* \mapsto z^* + \zeta, \; X_i \mapsto X_i + a_i.$$

  - Note that the first reduction modifies X but the second leaves Y fixed.

Remarks

- Key point: can show that we find closest cusp.
- Once in a cuspidal neighbourhood reduce in constant time.
- The hard part is to find the closest cusp.
- Elliptic points are on the boundary, i.e. can have more than one "closest" cusp.
- The fundamental domain we use is a union of cuspidal domains with boundaries of the form
    - compact × "wedge" close to the cusps, and
    - a union exterior of surfaces of the form $S_\lambda = \{\mathrm{N}(cz + d) = 1\}$ where $\lambda = (c : d)$.
- The "bottom" is complicated (this is where all relations in Γ_K show up).

Finding the closest cusp

- Let $z \in \mathbb{H}_K$ and $\lambda = \frac{a}{c} \in \mathbb{P}^1(K)$.
- Then
$$\Delta(z,\lambda)^2 = \mathrm{N}(y)^{-1}\,\mathrm{N}\left((-cx+a)^2 + c^2y^2\right).$$
- For each $r > 0$ there is only a finite (explicit!) number of pairs $(a',c') \in \mathcal{O}_K^2/\mathcal{O}_K^\times$ s.t.
$$\Delta(z,\lambda') \leq r.$$
- In fact, for $i = 1,\ldots,n$ we have bounds on each embedding:
$$\begin{aligned} |\sigma_i(c)| &\leq c_K r^{\frac{1}{2}} \sigma_i\left(y^{-\frac{1}{2}}\right), \\ |\sigma_i(a-cx)|^2 &\leq \sigma_i\left(rc_K^2 y - c^2y^2\right) \end{aligned}$$
- Here $c_K = r_K^{\frac{n-1}{2}}$ (an explicit constant).

Explicit bounds

The key to the proofs that the algorithms terminate are explicit versions of the following lemmas:

Lemma

There exists a constant $C_K > 0$ *s.t. if* $x \in \mathbb{R}^n$ *and* $\varepsilon > 0$ *then there are integers* $c, d \in \mathcal{O}_K$, $c \neq 0$,

$$\|cx + d\|_\infty \leq \varepsilon \text{ and } \|c\| \leq \frac{C_K}{\varepsilon}.$$

Lemma

There exists a constant $r_K > 0$ *s.t. if* $\alpha \in K$ *with* $\mathrm{N}\alpha = 1$ *then there exists* $\varepsilon \in \mathcal{O}_K^\times$ *such that*

$$|\sigma_i(\alpha\varepsilon)| \leq r_K^{\frac{n-1}{2}}.$$

Choice of constants

Proposition

We can take $C_K = 2^{\frac{1}{n}} \left(covol(\mathcal{O}_K)\right)^{\frac{2}{n}}$ *and*

$$r_K = \max_k \left\{ \frac{\max\left(\left|\sigma_1\left(\varepsilon_k\right)\right|, \ldots, \left|\sigma_n\left(\varepsilon_k\right)\right|, 1\right)}{\min\left(\left|\sigma_1\left(\varepsilon_k\right)\right|, \ldots, \left|\sigma_n\left(\varepsilon_k\right)\right|, 1\right)} \right\}.$$

Remark

$r_K \geq 1$ always. If $K = \mathbb{Q}\left(\sqrt{D}\right)$ has a fundamental unit ε_0 with $\sigma_1\left(\varepsilon_0\right) > 1 > \sigma_2\left(\varepsilon_0\right)$ then $r_K = \left|\sigma_1\left(\varepsilon_0\right)\right|^2$.

Example $\mathbb{Q}\left(\sqrt{5}\right)$

- The orders which can appear are: 3, 4, 5, 6, 8, 10, 12
- The possible traces are:

m	t	
3	-1	
4	0	
5	$\frac{1}{2}\left(\sqrt{5}-1\right)$	$\frac{1}{2}\left(-\sqrt{5}-1\right)$
6	1	
8	-	
10	$\varepsilon_0 = \frac{1}{2}\left(\sqrt{5}+1\right)$	$\varepsilon_0^* = \frac{1}{2}\left(-\sqrt{5}+1\right)$
12	-	

Example (contd.)

A set of reduced fixed points is:

order	trace	fixed pt	ell. matrix
4	0	i	$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
4	0	$i\varepsilon_0^*$	$SE(\varepsilon^*) = \begin{pmatrix} 0 & \varepsilon_0^* \\ -\varepsilon_0^* & 0 \end{pmatrix}$
6	1	ρ	$TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
6	1	$\rho\varepsilon_0^*$	$SE(\varepsilon_0) T^{\varepsilon^3} = \begin{pmatrix} 0 & \varepsilon_0^* \\ \varepsilon_0 & 1 \end{pmatrix}$
10	ε	$-\frac{1}{2}\varepsilon_0 + \frac{i}{2}\sqrt{3-\varepsilon_0}$	$ST^{\varepsilon_0} = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon_0 \end{pmatrix}$
10	ε^*	$\frac{1}{2}\varepsilon_0 + \frac{i}{2}\varepsilon_0^*\sqrt{3-\varepsilon_0^*}$	$T^{\varepsilon_0^*} S = \begin{pmatrix} \varepsilon_0^* & -1 \\ 1 & 0 \end{pmatrix}$

Here $\rho^3 = 1$ and we always choose "correct" Galois conjugates to get points in $\mathbb{H}^n$.

Example $\mathbb{Q}\left(\sqrt{3}\right)$

	t	z_t	$\frac{1}{\sqrt{\mathrm{N}y}}$	Y	X_1	X_2		
$4a$	0	$\frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}$	$\sqrt{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	
4	0	$\frac{-1+\sqrt{3}}{2} + i\frac{1-\sqrt{3}}{2}$	$\sqrt{2}$	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\sim 4a$
$4b$	0	$\varepsilon_0 i$	1	$-\frac{1}{2}$	0	0	0	
$4c$	0	i	1	0	0	0	0	
6	1	$\frac{1}{2} - i\left(1+\frac{\sqrt{3}}{2}\right)$	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\sim 12a$
$6a$	1	$\frac{1}{2}+\frac{1}{2}i\sqrt{3}$	$\sqrt{\frac{4}{3}}$	0	$\frac{1}{2}$	0	0	
$6b$	1	$\frac{\sqrt{3}}{2} - i\left(\frac{1}{\sqrt{3}}+\frac{1}{2}\right)$	$\sqrt{\frac{4}{3}}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	
$12a$	$-\sqrt{3}$	$\frac{1}{2}\sqrt{3}+\frac{1}{2}i$	2	0	0	$-\frac{1}{2}$	0	

Example $\mathbb{Q}(\sqrt{10})$ order 4

We have two cusp classes: $c_0 = \infty = [1:0]$ and $c_1 = \left[3 : 1+\sqrt{10}\right]$

Orders: 4 (trace 0) and 6 (trace 1).

order	label	fixed pt	close to
4	$4a$	$\left(\frac{1}{2}\sqrt{10}+\frac{3}{2}\right)\sqrt{-4}^{\pm}$	∞
4	$4b$	$\frac{1}{2}\sqrt{-4} = i$	∞
4	$4c$	$\left(\frac{1}{4}\sqrt{10}-\frac{3}{4}\right)\sqrt{-4}^{\pm}+\frac{1}{2}$	∞
4	$4d$	$\frac{1}{2}\sqrt{10}-\frac{1}{2}+\frac{1}{4}\sqrt{-4}$	∞
4	$4e$	$\frac{5}{13}\sqrt{10}-\frac{1}{2}+\frac{1}{52}\sqrt{-4}$	c_1
4	$4f$	$\frac{129}{370}\sqrt{10}-\frac{86}{185}+\left(-\frac{3}{740}\sqrt{10}+\frac{1}{185}\right)\sqrt{-4}^{\pm}$	c_1

Here $\sqrt{-4}^{\pm} = \pm 2i$ with sign chosen depending on the embedding of $\sqrt{10}$.

Example $\mathbb{Q}(\sqrt{10})$ order 4

label	x	$N(x)$	y	$N(y)$
$4a$	0	0	$\sqrt{10}-3$	-1
$4b$	0	0	-1	1
$4c$	$2\sqrt{10}+6$	-4	$2\sqrt{10}+6$	-4
$4d$	$-2\sqrt{10}+2$	-36	-2	4
$4e$	$-20\sqrt{10}+26$	-3324	-26	676
$4f$	-86	7396	$-15\sqrt{10}-20$	-1850

Note that if A is the cusp normalizing map of c_1 then

label	$A^{-1}z$	x	y		
$4e$	$\left(-\frac{1}{9}\sqrt{10}-\frac{7}{18}\right)\sqrt{-4}$	0	7		
$4f$	$\left(\frac{-1}{36}\sqrt{10}+\frac{1}{36}\right)\sqrt{-4}^{\pm}+\frac{1}{2}$	$-2\sqrt{10}-2$	$-2\sqrt{10}-2$		

Example $\mathbb{Q}(\sqrt{-10})$

Factoring matrices

Given elliptic element A:

- Find fixed point z
- Set $z_0 = z + \varepsilon$ s.t. $z_0 \in \mathcal{F}_\Gamma$ (well into the interior).
- $w_0 = Az_0$
- Find pullback of w_0 in to $\mathcal{F}_\Gamma$ (make sure $w_0^* = z_0$).
- Keep track of matrices used in pullback.

Example

$$K = \mathbb{Q}(\sqrt{3}),\ z = \frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}\ A = \begin{pmatrix} -1 & -\sqrt{3}+1 \\ \sqrt{3}+1 & 1 \end{pmatrix}$$

- $w_0 = Az_0 \sim$ (close to 0)
- $w_1 = Sw_0 \sim$ (close to $a - 1$)
- $w_2 = ST^{1-a}w_1$
- $w_3 = T^{1+a}w_2$ – reduced
- $A = T^{1+a}ST^{a-1}S$ (as a map)
- $A = S^2T^{1+a}ST^{a-1}S$ (in $\mathrm{SL}_2(\mathcal{O}_K)$)